HEAT TRANSFER IN CHANNEL FLOW OF A MICROPOLAR FLUID

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ABSTRACT

The study of heat transfer in channel flow has been done by previous authors for Newtonian and elastico-viscous fluids. It is the aim of the present paper to study the temperature profile for flow of a micropolar fluid in a channel induced by a constant axial pressure gradient, when the walls are maintained at constant temperatures. We have examined the effects of microrotation on the temperature profile and on the kinetic energy of the fluid. Three cases have been chosen by us for detailed study: (i) both the walls are maintained at different constant temperatures, (ii) both the walls are maintained at the same constant temperature, (iii) one wall is kept at a constant temperature and there is no heat flux at the other wall.

1. INTRODUCTION

The equations determining the velocity field and temperature field within a fluid in motion are, in general, interrelated. However, when buoyancy forces are disregarded and when the properties of the fluid may be assumed to be independent of the temperature, the velocity field does not depend on the temperature field, while the temperature field does depend on the velocity field. One of us² (R. R.) has recently studied the problem of velocity distribution in steady pressure gradient flow in a channel of a micropolar fluid. The aim of the present paper is to study the process of heat transfer in the above flow when the walls are maintained at constant temperature. We have examined the effects of microrotation on the temperature profile and have compared the results with those of forced convection in Newtonian fluids.³ Previous authors⁴–⁶ have investigated the nature of the temperature distribution in the above type of flow of several elastico-viscous non-Newtonian fluids and have studied in particular the effects of elasticity on the flow.
We use the energy equation of micropolar fluids given by Eringen in which the presence of coefficients of viscosity and gyroviscosity account for the heat generated by frictional and microrotational effects. Moreover, the presence of microrotation modifies the kinetic energy of the fluid.

In the numerical work, we have taken a number of values of the product of the Prandtl and Eckert number corresponding to both moderate and high velocity distributions in the following cases:

(i) both the plates are kept at different constant temperatures,

(ii) both the plates are kept at same constant temperature,

(iii) one plate is at a constant temperature and the heat flux from the other plate into the fluid is prescribed.

2. FORMULATION OF THE PROBLEM

We choose a rectangular Cartesian system with the axis of $x$ in the direction of the flow and the two plates are given by $y = h$ and $y = -h$. In the case of incompressible flow, the equations determining the velocity and temperature of the fluid are:

\[ v_{k,k} = 0, \quad (2.1) \]

\[ -p_{,k} + (\lambda_v + \mu_v) v_{l,k,l} + (\mu_v + K_v) v_{k,l} + K_v \varepsilon_{klm} v_{m,l} \]

\[ + \rho (f_k - \dot{v}_k) = 0, \quad (2.2) \]

\[ (\alpha_v + \beta_v) v_{l,k,l} + \gamma_v v_{k,l} + K_v \varepsilon_{klm} v_{m,l} - 2K_v v_k \]

\[ + \rho (l_k - j\dot{v}_k) = 0, \quad (2.3) \]

and

\[ \rho \varepsilon = -pd_{kk} + \lambda_v d_{ll}d_{kk} + (2\mu_v + K_v) d_{ll}d_{kk} \]

\[ + 2K_v (\omega_k - v_k) (\omega_k - v_k) + \alpha_v v_{k,l}v_{k,l} \]

\[ + \beta_v v_{k,l}v_{k,l} + \gamma_v v_{k,l}v_{k,l} - q_{k,k} + \rho h, \quad (2.4) \]

where $\lambda_v, \mu_v, K_v$ are coefficients of viscosity, $\alpha_v, \beta_v, \gamma_v$ are coefficients of gyroviscosity, $\rho$ is the density and $j$ is a constant on the assumption of micro-isotropy of the fluid, $(v_k), (v_{k,l})$ are the velocity and the microrotation vectors respectively, $\varepsilon$ is the internal energy density per unit mass, $(q_{k,k})$ is the heat
flux vector and \( h \) is the heat source per unit mass, \((d_{kl})\) is the deformation rate tensor and \((\omega_m)\) is the vorticity vector given by

\[
\omega_{kl} = - \varepsilon_{klm} \omega_m,
\]

\[
= \frac{1}{2} \left( \psi_k, l - \psi_l, k \right),
\]

(2.5)
a suffix following a comma denoting covariant differentiation.

The components of velocity and microrotation in the case of channel flow induced by a constant pressure gradient \( dp/dx \) have been determined earlier.\(^2\) Choosing

\[
v_1 = \tilde{u}(\tilde{y}), \quad v_2 = 0, \quad v_3 = 0,
\]

\[
v_1 = 0, \quad v_2 = 0, \quad v_3 = \tilde{v}(\tilde{y}),
\]

(2.6)
and non-dimensionalising according to the following scheme:

\[
\tilde{u} = u_0 u, \quad \tilde{y} = hy, \quad \tilde{v} = \frac{v u_0}{h}, \quad \frac{\mu v h^2}{c} = n_2,
\]

\[
\frac{K_v h^2}{c} = n_3, \quad u_0 = - \frac{h^2 \frac{dp}{dx}}{2 \mu v + \mu_0},
\]

(2.7)
we obtain

\[
u = 1 - y^2 - \frac{n_3}{n_2 + n_3} \cdot \frac{1}{a} \cdot \frac{\cosh \alpha - \cosh ay}{\sinh \alpha},
\]

\[
v = y - \frac{\sinh ay}{\sinh \alpha},
\]

where

\[
a^2 = \frac{n_3 (2 n_2 + n_3)}{n_2 + n_3}
\]

(2.8)
\( u \) and \( v \) satisfy the boundary conditions of “No slip” and “No micro-slip” at the channel wall, namely

\[
u = 0 \text{ on } y = \pm \frac{1}{2}, \quad v = 0 \text{ on } y = \pm \frac{1}{2},
\]

(2.9)
In the presence of this velocity field, the equation determining the temperature is given by

\[
\frac{d^2 T}{dy^2} + \frac{E \sigma}{2} \left( \frac{du}{dy} \right)^2 + \frac{2n_3}{n_2 + n_3} E \sigma \left( \frac{1}{2} \frac{du}{dy} + \nu \right)^2 \\
+ \frac{E \sigma}{2n_2 + n_3} \left( \frac{d\nu}{dy} \right)^2 = 0, \tag{2.10}
\]

where \( T \) is the non-dimensional temperature, \( E \) the Eckert number and \( \sigma \) the Prandtl number suitably defined.

**Case 1**

Both the walls are maintained at different constant temperatures (the upper wall at higher temperature). The boundary conditions in this case are:

\[
\bar{T} = \bar{T}_L \quad \text{on} \quad \bar{y} = -h, \quad \bar{T} = \bar{T}_U \quad \text{on} \quad \bar{y} = h. \tag{2.11}
\]

Non-dimensionalising the temperature by the following relation:

\[
T = \frac{\bar{T} - \bar{T}_L}{\bar{T}_U - \bar{T}_L},
\]

and defining the Prandtl and Eckert numbers by

\[
E = \frac{\mu_0^2}{C_p (\bar{T}_U - \bar{T}_L)}, \quad \sigma = \frac{C_p}{k} (2\mu_v + K_v),
\]

we now have to solve the equation (2.10) subject to boundary conditions

\[
T = 0 \quad \text{on} \quad y = -1, \quad T = 1 \quad \text{on} \quad y = 1. \tag{2.12}
\]

This gives us the temperature distribution

\[
T = \frac{1}{2} (y + 1) + \frac{E \sigma}{6} (1 - y^4) + \frac{E \sigma}{2n_2 + n_3} F(y), \tag{2.13}
\]

where

\[
F(y) = - \left( \frac{y^2 + 3}{2} \right) + \frac{2}{\alpha \sinh \alpha} (\alpha y \sinh \alpha y + \cosh \alpha - \cosh \alpha y) + \frac{1}{4 \sinh^2 \alpha} (\cosh 2\alpha - \cosh 2\alpha y). \tag{2.14}
\]
The first two terms give the temperature distribution same as the Newtonian fluid with viscosity \((2\mu_0 + K_0)\), while \(F(y)\) takes account of the microrotation of the fluid.

From (2.13), we have

\[
\frac{dT}{dy} \bigg|_{y=-1} = \frac{1}{2} - E\sigma \left\{ 2 - \frac{a \coth a - 1}{2n_2 + n_3} \right\},
\]

and

\[
\frac{dT}{dy} \bigg|_{y=1} = \frac{1}{2} + E\sigma \left\{ 2 - \frac{a \coth a - 1}{2n_2 + n_3} \right\},
\]

(2.15)

from which we deduce for a viscous fluid without microrotation, the following expressions for the temperature gradients at the walls:

\[
\frac{dT}{dy} \bigg|_{y=-1} = \frac{2}{3} \left( 3 - E\sigma \right), \quad \frac{dT}{dy} \bigg|_{y=1} = \frac{2}{3} \left( 3 + E\sigma \right).
\]

From the first relation (2.15'), it is clear that heat flows into or from the upper plate into the fluid according as

\[ E\sigma > 0 \quad \text{or} \quad < 0. \]

Hence if we fix \(T_U\) and \(T_L\), there exists a critical value

\[ u_{oc} = \left[ \frac{3K}{4(2\mu_0 + K_0)} (T_U - T_L) \right]^\frac{1}{2} \]

of \(u_o\) such that when \(u_o < u_{oc}\), the heat will flow from the upper plate to the fluid, while the reverse happens when \(u_o > u_{oc}\). From the second relation in (2.15'), we find that the fluid always imparts heat to the lower plate to maintain it at a constant temperature \(T_L\).

The total heat flux from the fluid to the walls

\[
= K \left( \frac{T_U - T_L}{h} \right) \left[ \frac{dT}{dy} \bigg|_{y=-1} - \frac{dT}{dy} \bigg|_{y=-1} \right],
\]

\[
= \frac{4}{3} E\sigma \left( \frac{T_U - T_L}{h} \right). \quad (2.16)
\]
This quantity is always positive and though, when \( E < 3/4 \), heat flows from the wall to the fluid, the total heat flux is always from the fluid to the wall. The heat generated by friction is given out to the walls.

For micropolar fluids, when \( dT/dy < 0 \) at the upper wall heat flows from the fluid to the wall. This occurs when

\[
E \sigma \left[ \frac{2 - \frac{\alpha \coth \alpha - 1}{2n_2 + n_3}}{3} \right] > \frac{1}{2}.
\]  

(2.17)

Since \( \frac{\alpha \coth \alpha - 1}{2n_2 + n_3} \) is always greater than zero, the effect of microrotation is to raise the critical value of \( E \sigma \) below which heat flows from the warmer wall to the fluid. That is, the critical value of \( u_0 \) is raised above which the heat begins to flow from the fluid to the warmer wall. The total heat flux from the fluid to the walls

\[
= K \frac{(\bar{T}_w - \bar{T}_l)}{h} E \sigma \left\{ \frac{2}{3} - \frac{\alpha \coth \alpha - 1}{2n_2 + n_3} \right\}.
\]  

(2.18)

Since \( \frac{\alpha \coth \alpha - 1}{(2n_2 + n_3)} \) is always less than \( 2/3 \) for all \( n_2 \) and \( n_3 \) the heat flux will be from the fluid to the wall, the cooler wall will always take heat from the fluid, while the warmer wall takes heat from the fluid when \( E \sigma \) exceeds a critical value. The total heat flux for a micropolar fluid is less than that for the corresponding Newtonian fluid with a coefficient of viscosity \( 2(\mu_v + K_v) \). This suggests that there is less dissipation of kinetic energy within a micropolar fluid and this results in the temperature at each point of a micropolar fluid being less than that for a Newtonian fluid.

We study next the kinetic energy (K.E.) of simple microfluid, which includes in it the contribution due to microrotation also. If \( V \) be a material volume of the fluid, the kinetic energy in \( V \) is given by

\[
\text{K.E.} = \frac{1}{2} \int_V \int_{\partial V} \rho' v_{k'} v_k' dv'.
\]  

(2.19)

Since

\[
v_{k'} = v_k + \nu_{k'\xi'\xi'};
\]  

(2.20)

\[
\text{K.E.} = \frac{1}{2} \int_V \rho v_k^2 dv + \frac{1}{2} \int_V \rho v_{k'\xi'\xi'}^s \xi' \xi' \, dv.
\]

where

\[
\int_{\partial V} \rho' dv = \rho dv, \quad \int_{\partial V} \rho' \xi' \xi' \, dv' = 0.
\]
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and

\[ \int_{V} \rho' \varepsilon' \varepsilon' \, dv' = \rho \delta \varepsilon \, dv; \]

\( \delta \varepsilon \) is a constant for the deformed body called the 'micro-inertia moments'.

In the case of a micropolar fluid

\[ i^{sl} = i^{os} = \frac{j}{2} \delta^{sl}, \quad \nu_{lk} = \epsilon_{mlk} \nu_{m}, \quad \nu_{sk} = \epsilon_{nsk} \nu_{n}, \]

\[ \therefore \quad \text{K.E.} = \frac{1}{2} \int_{V} \rho \nu_{k}^{2} \, dv + \frac{j}{2} \int_{V} \rho \nu_{k}^{2} \, dv. \quad (2.21) \]

In the present case, we have

\[ \text{K.E.} = \frac{1}{2} \rho u_{s}^{2} a \left[ \frac{16}{15} - \frac{8n_{3}}{n_{2} + n_{3}} \left\{ \coth \alpha \left( \frac{1}{3}a \right) - \frac{a}{3} \right\} + \frac{1}{a} \right] \]

\[ + \frac{n_{3}^{3}}{(n_{2} + n_{3})^{2}} \left\{ \frac{3 \coth^{2} \alpha - 1}{a^{2}} - \frac{3 \coth \alpha}{a^{2}} \right\} \]

\[ + 2j_{0} \left\{ \frac{5}{6} + \frac{2}{a^{2}} - \frac{3}{2a} \coth \alpha - \frac{1}{2} \coth^{2} \alpha \right\}. \quad (2.22) \]

Numerical computations, performed for micropolar fluids with \((n_{2} + n_{3})j_{0} = 4n_{3}/a^{2}\), show that the kinetic energy for this class of micropolar fluids is less than that for the Newtonian fluids. Moreover, from the first equation in (2.8), we conclude that at each location the fluid velocity for these micropolar fluids is less than that for the Newtonian fluids.

**Case 2**

Both the walls are maintained at the same temperature \( \bar{T}_{e} \). Non-dimensionalising the temperature \( \bar{T} \) by

\[ T = \frac{\bar{T} - \bar{T}_{e}}{\bar{T}_{e}} , \]

and defining

\[ E = \frac{u_{0}^{2}}{C_{p} \bar{T}_{e}}, \quad \sigma = \frac{C_{p}}{K} (2\mu_{D} + K_{D}), \quad (2.23) \]
we solve equation (2.10) subject to boundary conditions

\[ T = 0 \text{ on } y = \pm 1. \]  

(2.24)

This gives

\[ T = \frac{E}{6} (1 - y^2) + \frac{E\sigma}{2n_2 + n_3} F(y), \]  

(2.25)

where \( F(y) \) is given by (2.10).

The first term gives the contribution of a Newtonian viscosity \((2\mu_0 + K_0)\). The second term being always negative (as noted by numerical computation). We notice here also that the temperature at each point of the fluid is less than that of the corresponding Newtonian fluid. Computing \(dT/dy\) at the two walls, we find that the fluid gives less heat to the walls than the Newtonian fluid. The total heat flux from the fluid to the walls in this case

\[ = 2E\sigma \left[ \frac{2}{3} - \frac{a \coth a - 1}{2n_2 + n_3} \right] \frac{K\tilde{T}_e}{h}. \]  

(2.26)

Case 3

There is no heat flux from the fluid to the lower wall, that is the lower wall is perfectly insulated against the heat flow. Let us maintain the upper wall at a constant temperature \( \tilde{T}_u \) and define the dimensionless temperature \( T \) by \( T = (\tilde{T} - \tilde{T}_u)/\tilde{T}_u \). Using the relations (2.23), and solving the equation (2.10), under the boundary conditions

\[ T = 0 \text{ at } y = 1, \]

\[ \frac{dT}{dy} = 0 \text{ at } y = -1, \]  

(2.27)

we obtain

\[ T = \frac{E\sigma}{6} (5 - 4y - y^2) + \frac{E\sigma}{2n_2 + n_3} \left\{ F(y) - (1 - y)(a \coth a - 1) \right\}. \]  

(2.28)
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Fig. 1 (a)

Fig. 1 (b)

NEWTONIAN
MICROPOLAR FLUID, $\nu = 0.35, P = 6/7, n = 0.1$

A2
Again $T$ is less at each point of a micropolar fluid than for the corresponding Newtonian fluid.

We have at the upper wall

$$\left. \frac{dT}{dy} \right|_{y=1} = 2E\sigma \left[ -\frac{2}{3} + \frac{a \coth a - 1}{2n_2 + n_3} \right], \quad (2.29)$$

so that there is heat flux at the upper wall. The heat flows from the fluid to the upper wall, the heat flux being less than that for a Newtonian fluid. The maximum temperature within the fluid occurs at the lower wall, and this adiabatic wall temperature is given by

$$T_A = 2E\sigma \left[ \frac{2}{3} - \frac{a \coth a - 1}{2n_2 + n_3} \right], \quad (2.30)$$

3. Discussion of Results

Case 1.—We have performed numerical calculations for two specific micropolar fluids whose parameters are given by $n_2 = n_3 = 2/3$ and $n_2 = 6/35, n_3 = 6/7$. The velocity profiles for these two fluids lie within the Newtonian parabolic profile so that the velocity at each point of these fluids is less than that for the corresponding Newtonian fluid. It is noticed that for given value of $E\sigma$, the temperature at each point of these fluids is also less than that for the corresponding Newtonian fluid. Besides, a higher value of $E\sigma$ is required before the fluid begins to give heat to the warmer wall, the critical value of $E\sigma$ for the Newtonian fluids is $3/4$, while for the above micropolar fluids it is $0.98$ and $1.232$ respectively. As the ratio $n_3/n_2$ increases for micropolar fluids, the temperature profile continuously recedes from the Newtonian profile and there is less dissipation of kinetic energy within the fluid. Figures 1 (a) and 1 (b) show the temperature profile for this case. We note that in the case of Newtonian fluids, there is a greater dissipation of K.E., more heat is generated and the warmer wall gets heat from the fluid for a smaller critical value of $u_0$, than for a micropolar fluid. In order to control the large generation of heat within a Newtonian fluid and to increase the critical value of $u_0$, additives may be added to the fluid to make it micropolar in nature.

Similar observations are noticed in Cases 2 and 3. Figures 2 and 3 show the temperature profile for these cases.
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Fig. 2

--- NEWTONIAN
--- MICROPOLAR FLUID, \( n_1 = 2/3, \alpha = 1, \beta = 1 \)
--- MICROPOLAR FLUID, \( n_1 = 6/35, \alpha = 6/7, \beta = 5 \)

Fig. 3

--- NEWTONIAN
--- MICROPOLAR FLUID, \( n_1 = 2/3, \alpha = 1, \beta = 1 \)
--- MICROPOLAR FLUID, \( n_1 = 6/35, \alpha = 6/7, \beta = 5 \)
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5. REFERENCES