MEIJER'S G-FUNCTION AND THE TEMPERATURE IN A NONHOMOGENEOUS BAR

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ABSTRACT

In this paper we have employed Meijer's G-function to solve a problem of the temperature in a nonhomogeneous bar and shown how Meijer's G-function may be found useful in solving many problems of applied mathematics.

1. INTRODUCTION

As an example of the use of Meijer's G-function in heat conduction we shall consider the problem of determining a function \( u(x, t) \), if \( u = f(x) \) when \( t = 0 \), where \( u(x, t) \) represents the temperature in a nonhomogeneous bar with ends at \( x = -1 \) and \( x = 1 \) in which the thermal conductivity is proportional to \( 1 - x^2 \), and if the lateral surface of the bar is insulated, the heat equation has the form [1, p. 197, (8)]

\[
\frac{\partial u}{\partial t} = b \frac{\partial}{\partial x} \left[ (1 - x^2) \frac{\partial u}{\partial x} \right],
\]

where \( b \) is a constant, provided the thermal coefficient \( c_0 \) is a constant [1, p. 17, Sec. 9]. The ends \( x = \pm 1 \) are also insulated because the conductivity vanishes there.

In what follows for sake of brevity \( a_r \) denotes \( a_1, \ldots, a_r \); \( \lambda \) is a positive integer and the symbol \( \triangle (\lambda, a) \) represents the set of parameters

\[
\frac{a}{\lambda}, \frac{a + 1}{\lambda}, \ldots, \frac{a + \lambda - 1}{\lambda}.
\]

In this paper we have considered

\[
\frac{a}{\lambda}, \frac{a + 1}{\lambda}, \ldots, \frac{a + \lambda - 1}{\lambda}.
\]

\[ u = f(x) = (1 - x)^{a_r} G^{m,n}_{p,q} \left[ x (1 - x)^{\lambda} \right]_{b_{a_r}}. \]
The following formula is required in the proof:

\[
\int_{-1}^{1} \frac{(1 - x)^{s} P_{m}(x) G_{r, \lambda}^{p, q}}{b_{s}} \left[ z (1 - x)^{\lambda} \right] b_{s} dx
\]

\[
= \frac{2^{p+q}}{\lambda} G_{r+s, \lambda}^{p+q, q+s} \left[ 2^{\lambda} z \left\{ \frac{\Delta_{p} \left( \lambda, -\sigma \right), a_{r}, \Delta_{q} \left( \lambda, -\sigma \right) \Delta_{q} \left( \lambda, -\sigma + m \right), b_{s}, \Delta_{p} \left( \lambda, 1 - \sigma - m \right) \right\} \right]
\]

(1.3)

where

\[ r + s < 2 (p + q), \quad |\arg z| < (p + q - \frac{1}{2} r - \frac{1}{2} s) \pi, \]

\[ \text{Re} \left( \sigma + \lambda b_{j} \right) > -1, \quad j = 1, 2, \ldots, p. \]

which follows from [3, p. 198, (3.2)].

2. The solution to be obtained is

\[
u (x, t) = \frac{2^{p}}{\lambda} \sum_{n=0}^{\infty} (2n + 1) G_{r+s, \lambda}^{p+q, q+s} \times \left[ 2^{\lambda} z \left\{ \frac{\Delta_{p} \left( \lambda, -\sigma \right), a_{r}, \Delta_{q} \left( \lambda, -\sigma \right) \Delta_{q} \left( \lambda, -\sigma + n \right), b_{s}, \Delta_{p} \left( \lambda, 1 - \sigma - n \right) \right\} \right] \times P_{n} (x) e^{-b_{n} (n+1) t}, \]

(2.1)

where

\[ r + s < 2 (p + q), \quad |\arg z| < (p + q - \frac{1}{2} r - \frac{1}{2} s) \pi, \]

\[ \text{Re} \left( \sigma + \lambda b_{j} \right) > -1, \quad j = 1, 2, \ldots, p. \]

Proof.—The solution of the problem as given in [1, p. 197, (8)] is

\[
u (x, t) = \sum_{n=0}^{\infty} A_{n} P_{n} (x) e^{-b_{n} (n+1) t} \]

(2.2)

If \( t = 0 \), then by virtue of (1.2), we have

\[
(1 - x)^{s} G_{r, \lambda}^{p, q} \left[ z (1 - x)^{\lambda} \left\{ a_{r} \right\} b_{s} \right] = \sum_{n=0}^{\infty} A_{n} P_{n} (x). \]

(2.3)
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Multiplying both sides of (2.3) by $P_m(x)$ and integrating with respect to $x$ from $-1$ to 1, we obtain

$$\int_{-1}^{1} (1 - x)^{\sigma} P_m(x) G_{\alpha, \beta}^{p, q} \left[ z (1 - x)^{\lambda} \right] \frac{dx}{b_\alpha^n} = \sum_{\alpha = 0}^{\infty} A_{\alpha} \int_{-1}^{1} P_m(x) P_n(x) dx. \quad (2.4)$$

Using (1.3) and the orthogonality property of Legendre polynomials [2, p. 277, (13) and (14)], we get

$$A_m = (2m + 1) \frac{2^\lambda}{\lambda} G_{r + 2\lambda, s + 2\lambda}^{p + \lambda, q + \lambda}$$

$$\times \left[ 2^\lambda \left[ \triangle (\lambda, -\sigma), a_r, \triangle (\lambda, -\sigma) \triangle (\lambda, -\sigma + \rho), b_\beta, \triangle (\lambda, -1 - \sigma - \rho) \right] \right], \quad (2.5)$$

where

$$r + s < 2 (p + q), \quad |\arg z| < (p + q - \frac{1}{2} r - \frac{1}{2} s)\pi, \quad \Re (\sigma + \lambda b_j) > -1, \quad j = 1, 2, \ldots, p.$$

Now with the help of (2.2) and (2.5) the solution (2.1) follows immediately.

On specializing the parameters, the G-function may be converted into Bessel functions, Legendre functions, and other higher transcendental functions [2, pp. 434–44]. Therefore, the $f(x)$ given in (1.2) is of general character and hence may encompass several cases of interest.

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REFERENCES

