UNSTEADY COUETTE FLOW OF A CONDUCTING FLUID BETWEEN TWO PARALLEL PLATES UNDER A TRANSVERSE MAGNETIC FIELD—II

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ABSTRACT

The effect of a sudden change in magnetic field and pressure gradient on a steady Couette flow of a viscous incompressible fluid between two parallel plates in the presence of a transverse magnetic field is considered. Expressions for the velocity of the fluid in the disturbed flow, the intensities of the electrical and magnetic fields are obtained in two cases when the plates are non-conducting and when they are perfectly conducting.

1. INTRODUCTION AND BASIC EQUATIONS

We propose to examine the transient flow caused by disturbing the steady state obtained in the preceding paper (hence after called as Part I) by a sudden change in magnetic field from $H_0$ to $aH_0$ and in pressure gradient from $P$ to $\beta P$. A similar problem has been discussed by Rathi (1963) for Hartmann flow. The equations (2.6) of Part I become for the disturbed flow,

$$\frac{\partial u}{\partial t} = \beta P + \frac{1}{R} \frac{\partial^2 u}{\partial y^2} + \frac{aM^2}{RR_m} \frac{\partial h}{\partial y},$$

$$\frac{aM^2}{RR_m} h \frac{\partial h}{\partial y} = - \frac{\partial p}{\partial y},$$

$$\frac{\partial h}{\partial t} = \frac{1}{R_m} \frac{\partial^2 h}{\partial y^2} + \alpha \frac{\partial u}{\partial y},$$

$$\frac{\partial h}{\partial y} = - R_m (au + e). \quad (1.1)$$

We suppose that the change occurs at $t = 0$, so that for $t \leq 0$ the flow is determined by the steady state solutions $u_0$, $h_0$ and $e_0$ for the velocity of the fluid, the induced magnetic field and the electrical intensity. We
seek the solutions of equations (1.1) subject to certain initial conditions at
$t = 0$ (the steady state solutions in Part I) and the boundary conditions for
$t > 0$ (see § 3 of Part I).

2. **Solution of the Equations**

We limit ourselves to determine $u$, $h$ and $e$ only, whence the pressure
$p$ can be determined by the second equation of (1.1). Taking the Laplace
transform of (1.1) subject to the initial conditions

$$
\frac{s\ddot{u}}{s} + \frac{1}{R} \frac{d^2\ddot{u}}{dy^2} + \frac{aM^2}{RRm} \frac{d\ddot{h}}{dy} + u_0,
$$

$$
\frac{s\ddot{h}}{s} = \frac{1}{Rm} \frac{d^2\ddot{h}}{dy^2} + \frac{a}{dy} \frac{d\ddot{u}}{dy} + h_0,
$$

$$
\frac{d\ddot{h}}{dy} = -R_m(a\ddot{u} + \ddot{e}).
$$

**(Case 1.)** The plates are non-conducting.

$u_0$, $h_0$ and $e_0$ are given by (4.25–4.27) of Part I. Eliminating $h$, we
get

$$
\frac{d^4\ddot{u}}{dy^4} - \{s(R + R_m) + a^2M^2\} \frac{d^2\ddot{u}}{dy^2} + s^2RR_m\ddot{u} = (\beta - a) PRR_m
$$

$$
+ \frac{1}{R} \frac{RX_1chM}{M} - \frac{X_1shM}{shM} + \frac{sPR^2R_m}{M} \coth M,
$$

where

$$
X_1 = aM^2R_m + M^2R - sRR_m.
$$

Solving this equation for $\ddot{u}$, we get

$$
\ddot{u} = c_1chmy + c_2shmy + c_3chmy + c_4shmy + \frac{1}{RX_1} \frac{h_0My}{MX} \frac{PR}{shM}
$$

$$
+ \frac{PR}{sM} \coth M - \frac{X_1}{X} \frac{shM}{shM} + \frac{\beta - a}{s^2} P,
$$

where $c_1$, $c_2$, $c_3$ and $c_4$ are arbitrary constants of integration and

$$
m = \frac{1}{2} \{\sqrt{a^2M^2} + (\sqrt{R} + \sqrt{R_m})^2 s
$$

$$
+ \sqrt{a^2M^2} + (\sqrt{R} - \sqrt{R_m})^2 s\},
$$

(2.5)
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\[ n = \frac{1}{2} \left\{ \sqrt{a^2M^2} + \left( \sqrt{R} + \sqrt{R_m} \right)^2 s \right\} - \sqrt{a^2M^2} + \left( \sqrt{R} - \sqrt{R_m} \right)^2 s, \]  
(2.6)

\[ X = M^4 (1 - a^2) - sM^2 (R + R_m) + s^2RR_m. \]  
(2.7)

Substituting \( \tilde{u} \) in (2.1)

\[ \tilde{h} = \frac{K_1RR_m}{aM^2} (c_1chm + c_2chm) + \frac{K_2RR_m}{aM^2} (c_3shm + c_4chm) \]

\[ + \frac{R_m}{sM} \coth M - \frac{PR_{m}X_2}{sM^2} - \frac{PR_m}{M^2} X \frac{shm}{shM} \]

\[ + \frac{R_mX_2chm}{MX \frac{shm}{shM}}. \]  
(2.8)

\( \tilde{e} \) is then given by

\[ - \tilde{e} = \frac{sR - m^2 + \alpha^2M^2}{aM^2} (c_1chm + c_2chm) + \frac{sR - n^2 + \alpha^2M^2}{aM^2} \]

\[ \times (c_3shm + c_4chm) + \left\{ \frac{PR_{chm}}{MX \frac{shm}{shM}} - \frac{shm}{Xshm} \right\} (aX_1 - X_2) \]

\[ + \frac{PR}{sM} \left( \alpha \coth M - \frac{1}{M} \right) + \frac{\alpha(\beta - \alpha)}{s^2} \]  
(2.9)

where \( K_1 \) and \( K_2 \) are defined by (4.10) of Part I and

\[ X_2 = aM^2R + M^2R_m - sRR_m. \]  
(2.10)

We determine the arbitrary constants \( c_1, c_2, c_3 \) and \( c_4 \) in (2.4, 2.8, 2.9) using the boundary conditions (4.11) of Part I.

\[ \tilde{u} = \frac{P}{\triangle_1} (F_1chm - G_1chm) + \frac{1}{\triangle_2} (G_2shm - F_2shm) \]

\[ + \frac{PRX_1chm}{MX \frac{chm}{shM}} + \frac{PR \coth M}{sM} \frac{X_1shm}{X \frac{shm}{shM}} + \frac{\beta - \alpha}{s^2}, \]  
(2.11)

\[ \tilde{h} = \frac{PRR_m}{aM^2\triangle_1} (K_1F_1chm - K_2G_1chm) + \frac{R_m}{sM} \coth M - \frac{PRR_my}{sM^2} \]

\[ + \frac{RR_m}{aM^2\triangle_2} (K_2G_2chm - K_1F_2chm) - \frac{PRR_mX_2shm}{M^2 X \frac{shm}{shM}} \]

\[ + \frac{R_mX_2chm}{MX \frac{shm}{shM}}, \]  
(2.12)
\[-\bar{\varepsilon} = \frac{P}{aM^2\Delta_1} \{(sR - m^2 + a^2M^2) F_1 \text{ch} h \nu \}
\]
\[-(sR - n^2 + a^2M^2) G_1 \text{ch} h \nu \]
\[+ \frac{1}{aM^2\Delta_2}\{ (sR - n^2 + a^2M^2) G_2 \text{h} n \}
\[-(sR - m^2 + a^2M^2) F_2 \text{sh} n \nu \}
\[+ \frac{PR}{MX} \{ \frac{\text{ch} My}{\text{ch} M} - \frac{\text{sh} My}{X \text{sh} M} \}(aX_1 - X_2)
\[+ \frac{PR}{sM} (a \coth M \frac{1}{M}) + \frac{\alpha (\beta - \alpha) P}{s^2}, \quad (2.13)\]

where

\[
F_1 = \frac{\alpha (X + sX_2)}{sX} \text{ch} h + \left\{ \frac{\beta - \alpha}{s^2} + \frac{R \coth M}{MsX} \right\} (X + sX_1) K_2 \text{sh} n,
\quad (2.14)\]

\[
G_1 = \frac{\alpha (X + sX_2)}{sX} \text{ch} m + \left\{ \frac{\beta - \alpha}{s^2} + \frac{R \coth M}{MsX} \right\} (X + sX_1) K_1 \text{sh} m, \quad (2.15)\]

\[
F_2 = \frac{X + sX_1}{sX} K_1 \text{h} n + \frac{aM}{R} \coth M \frac{X + sX_2}{sX} \text{sh} n, \quad (2.16)\]

\[
G_2 = \frac{X + sX_1}{sX} K_1 \text{ch} m + \frac{aM}{R} \coth M \frac{X + sX_2}{sX} \text{sh} m, \quad (2.17)\]

and \(\Delta_1, \Delta_2\) are defined by (4.15, 4.16) of Part I.

The functions \(\bar{u}, \bar{h}\) and \(\bar{\varepsilon}\) have poles at \(s = o, s'\) and \(s''\).

Writing

\[
s = -\frac{a^2M^2 + \theta^2}{R (1 + r)^2}, \quad (2.18)\]

\(s'\) and \(s''\) are determined by roots \(\theta\) of equations (4.17, 4.19) of Part I where

\[
\phi = \sqrt{q^2\theta^2 -(1 - q^2)a^2M^2}. \quad (2.18)\]
X = 0 gives removable singularities. On inverting \( \tilde{u}, \tilde{h} \) and \( \tilde{e} \)

\[
\begin{align*}
\begin{aligned}
\nu & = \nu_{\text{stat.}} + \sum_{\nu'} \frac{4\text{Pr} \theta \phi^{st}}{Y' (1 + r)^3 s} \left[ \frac{a (X + sX_2)}{X} (\text{chmchmy} - \text{chnchmy}) \\
& \quad - \left\{ \frac{\beta - \alpha}{s} + \frac{R \coth M X}{MX} (X + sX_1) \right\} \{(K_1 \text{chnchmy} \\
& \quad - K_2 \text{chnchmy}) \right\} \right] - \sum_{\nu'} \frac{4r \theta \phi^{st}}{s(1 + r)^3 XY^s} \\
& \times \left\{ (X + sX_1) (K_1 \text{chnmshny} - K_2 \text{chnmshny}) \\
& \quad - \frac{aM}{R} \coth M (X + sX_2) (\text{shmshny} - \text{shmshny}) \right\}, \quad (2.19)
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
\begin{aligned}
h & = h_{\text{stat.}} + \sum_{\nu'} \frac{4\text{Pr}^2 \theta \phi^{st}}{(1 + r)^3 sY} \left[ \frac{X + sX_2}{XM^2} (K_1 \text{chnshmy} - K_2 \text{chnshmy}) \\
& \quad - \frac{a}{R^2} \left\{ \frac{\beta - \alpha}{s} + \frac{R \coth M X}{MX} (X + sX_1) \right\} \\
& \quad \times (\text{shmshny} - \text{shmshny}) \right] - \sum_{\nu'} \frac{4r^2 \theta \phi^{st}}{(1 + r)^3 Y^s X} \\
& \times \left[ (X + sX_1) a (\text{chnchmy} - \text{chnmch}) \\
& \quad - \frac{R r \coth M}{M} (X + sX_2) (K_1 \text{chnmshny} - K_2 \text{chnmshny}) \right], \quad (2.20)
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
ee & = e_{\text{stat.}} - \sum_{\nu'} \frac{4\text{Pr} \theta \phi^{st}}{aM^2 (1 + r)^3 sY'} \left[ \frac{a (X + sX_2)}{X} \\
& \times \left\{ (sR - m^2 + a^2 M^2) \text{chncl my} - (sR - n^2 + a^2 M^2) \text{chnmch} \right\} \\
& \quad + \left\{ \frac{\beta - \alpha}{s} + \frac{R \coth M X}{MX} (X + sX_1) \right\} \\
& \quad \times (sR - m^2 + a^2 M^2) K_2 \text{chnchmy} \\
& \quad - (sR - n^2 + a^2 M^2) K_1 \text{shmchny} \right] \\
& \times (sR - n^2 + a^2 M^2) K_1 \text{shmchny} \right] \\
& \times (sR - n^2 + a^2 M^2) K_1 \text{shmchny} \right], \quad (2.21)
\end{align*}
\]
where $Y'$ and $Y''$ are defined by (4.21, 4.22) of Part I. The values $u_{\text{stat.}}$, $h_{\text{stat.}}$ and $e_{\text{stat.}}$ in (2.19–2.21) which correspond to stationary conditions are calculated as residues for $s = 0$ and are determined by formulas

\begin{equation}
 u_{\text{stat.}} = \frac{\beta PR}{\alpha M} \left( \chi aM - \chi aMy \right) - \frac{\text{sh} aMy}{\text{sh} aM},
\end{equation}

\begin{equation}
 h_{\text{stat.}} = \frac{\beta PR M}{\alpha M^2 \text{sh} aM} \left( \text{sh} aMy - \nu \text{sh} aM \right) + \frac{R_m}{M^2} \left( \chi aM - \chi aMy \right),
\end{equation}

\begin{equation}
 e_{\text{stat.}} = \frac{PR}{\alpha M^2} \left( \alpha M \text{coth} \alpha M - 1 \right).
\end{equation}

**Case ii.**—The plates are perfectly conducting.

$u_0$ and $h_0$ are given by (4.40, 4.41) of Part I. $\bar{u}$ satisfies

\begin{equation}
 \frac{d^4 \bar{u}}{dy^4} - \{s (R + R_m) + \alpha^2 M^2\} \frac{d^2 \bar{u}}{dy^2} + s^2 RR_m \bar{u}
 = (\beta - \alpha) \frac{PR}{R_m^2 M^2} + \frac{PRX_1}{M^2 \text{ch} M} \text{ch} My - \frac{X_1 \text{sh} My}{\text{sh} M}.
\end{equation}

Solving for $\bar{u}$

\begin{equation}
 \bar{u} = c_1 \text{ch} My + c_2 \text{sh} My + c_3 \text{ch} My + c_4 \text{sh} My + \frac{(\beta - \alpha) P}{s^2} + \frac{PR}{M^2 s}
 + \frac{PRX_1}{M^2 \text{ch} M} \text{ch} My - \frac{X_1 \text{sh} My}{\text{sh} M},
\end{equation}

(2.26)
whence, from (2.1)
\[
\tilde{h} = \frac{K_1RR_m}{aM^2} (c_3shm + c_4chmy) + \frac{K_2RR_m}{aM^2} (c_3shm + c_4chmy) \\
- \frac{PRR_m y}{sM^2} - \frac{PRR_m}{M^3} \frac{X_2shm}{XchM} + \frac{R_m}{sM^2} + \frac{R_m}{M} \frac{X_2chM}{X shm}.
\]

Determining the constants \(c_1, c_2, c_3\) and \(c_4\) by (4.28) of Part I,
\[
\tilde{u} = - \frac{PR}{\Delta_3} (F_3chlmy - G_3chlmy) + \frac{R}{\Delta_4} (F_4chlmy - G_4chlmy) \\
+ \frac{PR}{sM^2} + \frac{PR}{M^3} \frac{X_1chM}{XchM} - \frac{X_1shm}{Xshm}, \quad (2.28)
\]
\[
\tilde{h} = - \frac{PR^2R_m}{aM^2 \Delta_3} (K_1F_3chlmy - K_2G_3chlmy) \\
+ \frac{R^2R_m}{aM^2 \Delta_4} (K_1F_4chlmy - K_2G_4chlmy) - \frac{PRR_m y}{sM^2} + \frac{R_m}{sM^2} \\
- \frac{PRR_m}{M^3} \frac{X_2shm}{XchM} + \frac{R_m}{MX} \frac{X_2chlM}{Xshm}, \quad (2.29)
\]
\[
\tilde{\epsilon} = \frac{PR}{aM^2 \Delta_3} \left\{ (a^2M^2 + sR - m^2) F_3chlmy - (a^2M^2 + sR - n^2) G_3chlmy \right\} \\
- \frac{R}{aM^2 \Delta_4} \left\{ (a^2M^2 + sR - m^2) F_4chlmy - (a^2M^2 + sR - n^2) \right\} \\
\times G_4chlmy \right\} - \frac{PR}{sM^2} (a - 1) - \frac{aP (\beta - a)}{s^2} + \frac{PR}{M^2} \frac{chlM}{chM} \\
- \frac{shM (X_2 - aX_1)}{shM} \frac{X}{X}, \quad (2.30)
\]

where
\[
F_3 = \left[ nK_2 \left\{ \frac{(\beta - a)}{s^2} + \frac{R}{M^2} \frac{X + sX_1}{sX} \right\} + \frac{a (X + sX_2)}{sX} \right] chn, \quad (2.31)
\]
\[
G_3 = \left[ mK_1 \left\{ \frac{(\beta - a)}{s^2} + \frac{R}{M^2} \frac{X + sX_1}{sX} \right\} + \frac{a (X + sX_2)}{sX} \right] chm, \quad (2.32)
\]
\[
F_4 = \left\{ nK_2 \frac{X + sX_1}{sX} + \frac{aM^2 (aX + sX_2)}{R sX} \right\} shn, \quad (2.33)
\]
\[
G_4 = \left\{ mK_1 \frac{X + sX_1}{sX} + \frac{aM^2 (aX + sX_2)}{R sX} \right\} shm, \quad (2.34)
\]
\( \Delta_a \) and \( \Delta_b \) are defined by (4.32, 4.33) of Part I.

The functions \( \tilde{u}, \tilde{h} \) and \( \tilde{e} \) have poles at \( s = o, s', s'' \) whilst

\[
X = o, \quad s = -\frac{a^2 M^2}{R \left(1 + r\right)^2}
\]

and

\[
s = -\frac{a^2 M^2}{R \left(1 - r\right)^2}
\]

give removable singularities.

Writing

\[
s = -\frac{a^2 M^2 + \theta^2}{R \left(1 + r\right)^2},
\]

\( s' \) and \( s'' \) are determined by

\[
\theta = \frac{-k\pi \pm \sqrt{q^2 k^2 \pi^2 - \left(q^2 - 1\right)^2 a^2 M^2}}{q^2 - 1},
\]

\( k \) assuming odd or even integral values according as \( s \) is \( s' \) or \( s'' \).

On inversion

\[
u = u_{\text{stat.}} + \sum \frac{4Pe^{st}t}{sZ'(1 + r)^2} \left[ \frac{\beta - \alpha}{s} + \frac{R}{M^2} \frac{X + sX_1}{X} \right]
\]

\[
\times (nK_{chmychn} - mK_{chmchny}) + \frac{a}{X} (X + sX_2)
\]

\[
\times (chmychn - chmchny) \bigg] - \sum \frac{4e^{st}}{sX(1 + r)^2Z''}
\]

\[
\times \left\{ (X + sX_1) (nK_{shmyshn} - mK_{shmshny}) \right. \\
\left. + \frac{aM^2}{R} (aX + sX_2) (shmyshn - shmshny) \right\},
\]

(2.35)
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\[ h = h_{\text{stat}} - \sum \frac{4P_{\text{rest}}^{\prime}}{sZ'(1 + r)^2} \left[ a \left( \frac{\beta - \alpha}{s} \right) + \frac{R}{M^2} \left( X + sX_1 \right) \right] \]

\[ \times (n \text{chnshmy} - m \text{cl my}) - \frac{R^2r}{M^2X} \]

\[ \times (X + sX_2) (K_1 \text{chm ychn} - K_2 \text{sh mychm}) \]

\[ + \sum \frac{4P_{\text{rest}}^{\prime}}{sX(1 + r)^2Z''} \left\{ \frac{X + sX_1 a (n \text{chnshmy} - m \text{cl my})}{R} \right\} \]

\[ - r (aX + sX_2) (K_1 \text{chm ychn} - K_2 \text{sh mychm}) \right\}, \quad (2.36) \]

\[ e = \sum \frac{4P_{\text{rest}}^{\prime}}{sR (1 + r)^2Z'} \left\{ \frac{\beta - \alpha}{s} + \frac{R}{M^2} \left( X + sX_1 \right) \right\} \left( n^2 \text{chnshmy} \right) \]

\[ - m^2 \text{chnchmny} \right) - \sum \frac{4P_{\text{rest}}^{\prime}}{sR \left(1 + r\right)^2Z'} (X + sX_2) \]

\[ \times \left\{ \left( a^2M^2 + sR - m^2 \right) \text{chnchmy} - (a^2M^2 + sR - n^2) \text{chmchmy} \right\} \]

\[ + \sum \frac{4e_{\text{rest}}^{\prime}}{(1 + r)^2Z''} \left[ - r^2 (s \text{nh smmy} - \text{sh mshmy}) + \frac{X_2}{RX} \right] \]

\[ \times \left\{ \left( a^2M^2 + sR - m^2 \right) \text{s l nshmy} - (a^2M^2 + sR - n^2) \text{s l mshmy} \right\} \]

\[ - \frac{aX_1}{RX} \left( n^2 \text{shnshmy} - m^2 \text{shnshmy} \right) \right\}, \quad (2.37) \]

where \( Z' \) and \( Z'' \) are defined by (4.37, 4.38) of Part I.

The values \( u_{\text{stat}} \) and \( h_{\text{stat}} \) in (2.35, 2.36) which correspond to stationary conditions are calculated as residues for \( s = o \) and are determined by the formulas

\[ u_{\text{stat}} = \frac{\beta PR}{a^2M^2 \text{chaM}} (\text{chaM} - \text{chaM}) + \frac{\text{shM} \text{y}}{\text{shM}}, \quad (2.38) \]

\[ h_{\text{stat}} = \frac{\beta P R R_m}{a^2M^2 \text{chaM}} (\text{shM} \text{y} - a \text{MchaM}) \]

\[ + \frac{R_m (\text{shM} - a \text{MchaM})}{aM^2 \text{shM}}. \quad (2.39) \]
3. Conclusions

From the solution it is clear that for small values of $R$, the transition is more rapid as compared to that for large values of $R$. The shape of the velocity profile does not change in transition state and the large part of the transition occurs in a small time. If the magnetic field is strong, the change is again rapid than otherwise. Again several of the first non-stationary terms in the solutions (2.19–2.21) and (2.35–2.37) represent damped oscillations for a given relation between the parameters. As $t \to \infty$, the flow is determined by the stationary conditions $u_{\text{stat.}}$, $h_{\text{stat.}}$, and $e_{\text{stat.}}$.

4. References