GREEN'S THEOREM IN PARACOMPACT MANIFOLDS MODELLED ON HILBERT SPACE

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RECENTLY it was shown by the author that Green's theorem which has been very useful in establishing certain results for compact spaces could be proved to hold good for manifolds which are not necessarily compact and metrizable. To prove the theorem, the affine connexion was suitably restricted.

In the present note it is shown that Green's theorem holds good for paracompact manifolds modelled on Hilbert space; consequently the integral formulae which are known to hold good for compact manifolds would also hold good for such paracompact manifolds. The author was motivated to do so in view of the fact that every compact manifold is paracompact whereas the converse is not true.

Preliminaries.—A topological space \( S \) is said to be paracompact if every open cover \( \{A\} \) of \( S \) admits of a subcover \( \{B\} \) such that

(i) It is a refinement of \( \{A\} \), i.e., every member of \( \{B\} \) is contained in some member of \( \{A\} \).

(ii) It is locally finite, i.e., every point of \( S \) has a neighbourhood intersecting finitely many members of \( \{B\} \).

A topological manifold \( \mathcal{M} \) is a Hausdorff space satisfying the second axiom of countability.

\( C^r \)-differentiable manifold \( \mathcal{M} \) modelled on Hilbert space is a topological manifold together with a differentiable structure \( \mathcal{D} \) of class \( C^r \).

Latin indices \( i, j, k \ldots \) take the values 1 to \( n \).

Greek indices \( \alpha, \beta, \gamma \ldots \) stand for members of \( J^n \).

215
is a collection of co-ordinate neighbourhoods \((U_a, h_a)\) covering \(\mathcal{M}\) and satisfying the following two conditions:

(1) For any two pairs \((U_a, h_a), (U_\beta, h_\beta)\) \(\in\mathcal{D}\) the homeomorphic map
\[ h_a h_\beta^{-1}: h_\beta(U_a \cap U_\beta) \to \text{open subsets of Hilbert space} \]
is \(C^r\) differentiable.

(2) \(\mathcal{D}\) is maximal with respect to the above property.

A manifold \(\mathcal{M}\) is said to be orientable if there exists at least one differentiable structure \(\mathcal{D}\) on it in which for every pair of co-ordinate neighbourhoods with non-empty intersection the mapping \(h_a h_\beta^{-1}\) is sense-preserving.

The differentiable manifold \(\mathcal{M}\) throughout this note is orientable paracompact, and is modelled on a Hilbert space. It is also assumed that transformation matrices carry convergent sequences into convergent ones.

\[ f, (m) = 1 \quad \text{for} \quad \| m \| \leq r_1 \]
\[ f, (m) = 0 \quad \text{for} \quad \| m \| \geq r_2 \]
where \(r_1\) and \(r_2\) are real numbers satisfying \(0 < r_1 < r_2\).

Since \(\mathcal{M}\) is paracompact, \(\sum_{a=1}^{\infty} h_a(m)\) is meaningful and the collection \(\{f_a(m)\}\) of functions defined as
\[ f_a(m) = \frac{h_a(m)}{\sum_{a=1}^{\infty} h_a(m)} \]
forms a partition of unity.
It is also known that if \( M \) is assumed to be connected a Riemannian metric can be trivially defined on \( M \).

We now state and prove the theorem locally and observe that condition (i) of differentiable structure ‘\( D \)’ ensures its global extension.

(For convenience, the dimension of \( M \) is taken to be \( n \).)

**Theorem:** In a paracompact orientable manifold \( M \), for any arbitrary vector field \( \lambda^i(m) \) we have

\[
\int_M \lambda^i, i \ dv = 0
\]

where

\[
dv = \sqrt{g} dx^1 A dx^2 \cdots dx^n
\]

(\( g = |g_{ij}| \) is Riemannian metric tensor determinant).

**Proof**—Let \( U = \{u_a\} \) denote the cover of \( M \) and a family \( F = \{f_a\} \) be a partition of unity, and let carrier \( f_a \) be denoted by \( \tilde{w}_a \), then since \( W = \{\tilde{w}_a\} \) is a refinement of \( U \), for each \( \tilde{w}_a \) there is a co-ordinate neighbourhood \( u_a \) say which contains \( \tilde{w}_a \).

Define

\[
\lambda^i_a = f_a \lambda^i.
\]  

(1)

Clearly

\[
\sum_{a=1}^{\infty} \lambda^i_a = \sum_{a=1}^{\infty} f_a \lambda^i = \lambda^i.
\]

Consequently

\[
\lambda^i, i = \sum_{a=1}^{\infty} \lambda^i, i.
\]

Now

\[
\int_M \lambda^i_a, i dv = \int_{M \sim \tilde{w}_a} \lambda^i_a, i dv + \int_{\tilde{w}_a} \lambda^i_a, i dv
\]

\[= I_1 + I_2.\]

In view of relation (1), \( \lambda^i_a \) vanishes throughout \( M \sim \tilde{w}_a \), consequently \( I_1 \) is zero.

Also \( \tilde{w}_a \) is a closed subregion of an orientable manifold, therefore integral taken around any contour in this region vanishes hence \( I_2 \) is zero.
Thus
\[ \int_M \lambda^i_{a,} idv = 0. \]

Since
\[ \int_M \lambda^i_{a,} idv = \sum_{a=1}^{\infty} \int_M \lambda^i_{a,} idv. \]

We have the required result.

REFERENCES