COMPARATIVE STUDY OF SOME CONSTITUTIVE EQUATIONS CHARACTERISING NON-NEWTONIAN FLUIDS

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Abstract

This paper compares, in a general way, the predictions of the constitutive equations given by Rivlin and Ericksen, Oldroyd, and Walters. Whether we consider the rotational problems in cylindrical co-ordinates or in spherical polar co-ordinates, the effect of the non-Newtonicity on the secondary flows is collected in a single parameter α which can be explicitly expressed in terms of the non-Newtonian parameters that occur in each of the above-mentioned constitutive equations. Thus, for a given value of α, all the three fluids will have identical secondary flows. It is only through the study of appropriate normal stresses that a Rivlin-Ericksen fluid can be distinguished from the other two fluids which are indistinguishable as long as this non-Newtonian parameter has the same value.

1. Introduction

Recently, a number of workers have investigated the secondary flows of non-Newtonian fluids, characterised by the constitutive equations given by (*a*) Rivlin and Ericksen (R. E.)[1], (*b*) Oldroyd (O)[2] and (*c*) Walters (W),[3] induced by the rotation of boundaries of different shapes, such as (i) an infinite plane about a normal, (ii) two parallel infinite planes about a common normal, (iii) one circular cylinder about its axis or (iv) two circular cylinders about their common axis, (v) one sphere about its diameter or (vi) two concentric spheres about a common diameter or (vii) two cones with common vertex about their common axis. For a given geometry of the boundary surface or surfaces, we find that the pattern of the secondary flow is similar whatever be the constitutive equation. This fact suggested that it would be useful to compare, in a general manner, the flow of the fluids characterised by various constitutive equations. Accordingly, we have considered in this note the general flows using cylindrical co-ordinates to cover up the cases (i) to (iv) and spherical polar co-ordinates to cover up the cases (v) to (viii).

We find that, in either case, the secondary flow is governed by a differential equation involving just one parameter α which can be explicitly expressed in terms of the non-Newtonian parameters occurring in these constitutive equations. Thus, these fluids for a given value of this parameter α have identical flows. The individual characteristics of a constitutive equation show up only in some normal stresses. Consequently, only the comparison of these normal stresses can help us in deciding whether a given real fluid corresponds to a specific constitutive equation.

For ready reference we record below the constitutive equations which we have mentioned above adopting the following notations:

$\mathrm{T} = [\mathrm{T}_{ij}]$ is the stress tensor,

$\mathrm{E} = [e_{ij}] = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the rate of strain tensor,

$u = [u_i]$ is the fluid velocity at $(\vec{r}, t)$

$a = [a_i] = \left(\frac{\partial u_i}{\partial t} + u^m u_{i,m}\right)$ is the acceleration vector at $(\vec{r}, t)$,

$\mathrm{D} = [\mathrm{D}_{ij}] = (a_{i,j} + a_{j,i} + 2u^m{}_{,i}\, u_{m,j})$ is the acceleration gradient tensor,

$\Omega = [\Omega_{ij}] = \frac{1}{2}(u_{j,i} - u_{i,j})$ is the vorticity tensor,

I is the Idem tensor,

δ_{ij} are Kronecker deltas. (1.1)

(*a*) ***Rivlin-Ericksen constitutive equation.***—Rivlin and Ericksen have given the following general constitutive equation:

$$\mathrm{T} = -p\mathrm{I} + 2\Phi_1\mathrm{E} + \Phi_2\mathrm{D} + \Phi_3\mathrm{E}^2 + \Phi_4\mathrm{D}^2 + \Phi_5(\mathrm{ED} + \mathrm{DE}) + \Phi_6(\mathrm{E}^2\mathrm{D} + \mathrm{DE}^2) + \Phi_7(\mathrm{ED}^2 + \mathrm{D}^2\mathrm{E}) + \Phi_8(\mathrm{E}^2\mathrm{D}^2 + \mathrm{D}^2\mathrm{E}^2) \tag{1.2}$$

which involves eight parameters Φ_i. However we have considered here its particular case

$$\mathrm{T} = -p\mathrm{I} + 2\eta\mathrm{E} + \Phi_2\mathrm{D} + 4\Phi_3\mathrm{E}^2 \tag{1.3}$$

obtained by putting $\Phi_i = 0$ when $i \geq 4$, where

η = coefficient of viscosity,

Φ_2 = coefficient of viscoelasticity,

and

$4\Phi_3$ = cross viscosity (we have taken $4\Phi_3$ and not Φ_3 to keep the same definition for E everywhere).

We note that the memory or elasticity of the fluid is incorporated in (1.3) through the acceleration-gradient tensor. The justification of taking this simplified version of the R-E constitutive equation lies in the fact that in the cases so far investigated, the retention of other terms does not alter the basic character of the flows.

(*b*) *The Oldroyd constitutive equation.*—Oldroyd has given the following general constitutive equation:

$$T_{ik} = S_{ik} - p\delta_{ik} \tag{1.4}$$

$$S^{ik} + \tau_1 \frac{\delta}{\delta t} S^{ik} + \mu_0 S_j^j E^{ik} - \mu_1 (S^{ij} E_j^k + S^{jk} E_j^i) + \nu_1 S^{jm} E_{jm} \delta^{ik}$$

$$= 2\eta \left[E^{ik} + \tau_2 \frac{\delta}{\delta t} E^{ik} - 2\mu_2 E^{ij} E_j^k + \nu_2 E^{jm} E_{jm} \delta^{ik} \right], \tag{1.5}$$

which also involves eight parameters. Here the convected derivative of a second order contravariant tensor B^{ik} is defined by

$$\frac{\delta}{\delta t} B^{ik} \equiv \frac{\partial}{\partial t} B^{ik} + u^j B^{ik}{}_{,j} + \Omega^k{}_{.m} B^{im} + \Omega^i{}_{.m} B^{mk}. \tag{1.6}$$

From the experimental data, Oldroyd concluded that the non-Newtonian behaviour of the elastico-viscous fluids, like high polymer solutions, can be explained on the basis of a simpler equation of state obtained from this general equation by putting

$$\mu_1 = \tau_1,\ \mu_2 = \tau_2,\ \nu_1 = \nu_2 = 0,\ \eta > 0,$$

$$\tau_1 > \tau_2 \geq \tfrac{1}{9}\tau_1 > 0,\ \mu_0 > 0. \tag{1.7}$$

Accordingly, we have also considered here the following simplified version of the Oldroyd equation of state:

$$S^{ik} + \tau_1\left(\frac{\delta}{\delta t} S^{ik} - S^{ij} E_j^{\ k} - S^{jk} E_j^{\ i}\right) + \mu_0 S_j^{\ j} E^{ik}$$

$$= 2\eta\left[E^{ik} + \tau_2\left(\frac{\delta}{\delta t} E^{ik} - 2E^{ij} E_j^{\ k}\right)\right], \tag{1.8}$$

where

τ_1 = stress relaxation time,

and

$\tau_2 (< \tau_1)$ = rate of strain retardation time.

Here the elasticity of the fluid is incorporated through the relaxation times τ_1 and τ_2.

(*c*) *The Walters Constitutive Equation.*—We have considered here the following form of the Walters constitutive equation:

$$S^{ik} = 2\int_0^\infty \frac{N(\tau)}{\tau} d\tau \int_{-\infty}^{t} \exp.\left(\frac{t' - t}{\tau}\right) \frac{\partial x^i}{\partial x'^m} \frac{\partial x^k}{\partial x'^r} E^{mr}(x', t')\, dt' \tag{1.9}$$

where

$N(\tau)\, d\tau$ = total viscosity of the Maxwell elements whose relaxation times lie between τ and $\tau + d\tau$,

(x') = co-ordinates of a fluid element at time t'

and

(x) = co-ordinates of the same fluid element at $t (> t')$.

2. Secondary Flows

The equations of state described in § 1 have been established entirely from mathematical standpoint and as such we cannot say *a priori* if any one of them represents a real non-Newtonian fluid. Moreover, the usual viscometers just measure the effective viscosity and we are unable to separate out the effects of various parameters that occur in each of these constitutive equations. These considerations led to the detailed study of the flows under various conditions using these constitutive equations to enable us to compare the theoretical predictions with the observations.

In order to simplify the mathematical analysis, the non-Newtonian effects are usually treated as of second order in comparison with the effects of the Newtonian viscosity. This assumption led to the concept of primary and secondary flows. We define the primary flows as the one that is realized by taking into account only the Newtonian viscosity under Stokes' assumption. We then define the secondary flow as the one that is superimposed on the primary flow due to inclusion of the inertial and non-Newtonian effects. Since the equations describing the secondary flows are highly complicated and non-linear, we take the inertia terms into account only in Oseen's approximation, and effect consequent simplification in writing the non-Newtonian terms also.

We can also define the primary and secondary flows in the following equivalent manner. We render the equations dimensionless and introduce an appropriate Reynolds number R. Then, if we expand the physical and dynamical variables in powers of R, we find that the zeroth order approximation corresponds to the primary flow while the first order approximation corresponds to the secondary flow. We have recorded this definition as it brings out clearly the physical situation under which our investigations are valid.

In the sequel we have used only the dimensionless variables. We take some appropriate length a and $a\Omega$, where Ω is an appropriate angular velocity of rotation, as characteristic length and characteristic velocity and then define the Reynolds number by

$$\mathrm{R} = \frac{a^2 \Omega \rho}{\eta}. \tag{2.1}$$

3. Cylindrical Co-ordinates

In this section we shall compare the steady secondary flows of the fluids mentioned above in cylindrical co-ordinates.

Primary flow.—In the primary flow, we have

$$u_0 = 0, \; v_0 = v_0(r, z), \; w_0 = 0 \tag{3.1}$$

where z-axis is taken along the axis of rotation, and v_0 is determined by

$$L^2[v_0] - \frac{v_0}{r^2} = 0, \quad L \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \tag{3.2}$$

with no slip condition at a solid boundary and regularity of the solution at infinity if the fluid extends to infinity. Here the non-vanishing stress components are:

$$S_{r\theta}^{(0)} = r\frac{\partial}{\partial r}\left(\frac{v_0}{r}\right), \quad S_{\theta z}^{(0)} = \frac{\partial v_0}{\partial z}. \tag{3.3}$$

Secondary flow.—We record in Table I the non-zero stress components in the secondary flow.

TABLE I

	R–E	0	W
$S_{rr}^{(1)}$	$2u_r + (2K + S)[S_{r\theta}^{(0)}]^2$	$2u_r$	$2u_r$
$S_{\theta\theta}^{(1)}$	$\frac{2u}{r} + S\,\bar{S}^{(0)}$	$\frac{2u}{r} + 2a\,\bar{S}^{(0)}$	$\frac{2u}{r} + 2a\,\bar{S}^{(0)}$
$S_{zz}^{(1)}$	$2w_z + (2K + S)[S_{\theta z}^{(0)}]^2$	$2w_z$	$2w_z$
$S_{rz}^{(1)}$	$(u_z + w_r) + (2K + S)\,S_{r\theta}S_{z\theta}^{(0)}$	$u_z + w_r$	$u_z + w_r$
a	$K + S = \frac{1}{\eta}(\Phi_2 + \Phi_3)\Omega$	$\Omega(\tau_1 - \tau_2)$	$\frac{\int_0^\infty \tau N(\tau)\,d\tau}{\int_0^\infty N(\tau)\,d\tau} \cdot \Omega$

where

$$\bar{S}^{(0)} = [S_{r\theta}^{(0)}]^2 + [S_{\theta z}^{(0)}]^2. \tag{3.4}$$

Introducing a stream function ψ through the relation

$$u = \frac{1}{r}\psi_z, \quad w = -\frac{1}{r}\psi_r \tag{3.5}$$

we have

$$L^2 [\psi] = R \left[-2v_0 \frac{\partial v_0}{\partial z} \right] + 2a \frac{\partial}{\partial z} \bar{S}^{(0)}. \tag{3.6}$$

Evidently ψ satisfies the following boundary conditions:

$$\psi_r = \psi_z = 0 \text{ on solid boundary}$$

and ψ_r, ψ_z remain bounded and regular when distance from the axis of rotation tends to infinity if the fluid extends to infinity.

We note the following important points:

(i) The secondary flow is in the planes passing through the axis of rotation and hence is orthogonal to the primary flow.

(ii) The effect of the non-Newtonicity on the stream function is collected in a single parameter a, so that all the three fluids should show identical secondary flows for a given value of a.

(iii) All the stress components are identical for the O and W fluids for a given value of a. The stress component for a R–E fluid differ from those for the O and W fluids due to the presence of terms containing (2K + S). Thus a R–E fluid for which 2K + S is either zero or very nearly equal to zero will behave more or less like a W or an O fluid.

(iv) In case of an O fluid we can in no way separate the effects of relaxation times in experiments using the cylindrical geometry and steady rotation. Similar remark is applicable for a W-fluid also. However, in case of a R–E fluid it is possible to evaluate K and S separately through the measurement of stresses on the boundary provided we can determine a by some characteristic property of the secondary flow, such as its separation and reversal.

(v) For the flow[4] induced by the rotation of one infinite plate or two parallel infinite plates, $v_0 = rw_0(z)$, so that $S_{r\theta}^{(0)} = 0$. Consequently, all the secondary stress components for these three fluids are identical except that the normal stress components $S_{\theta\theta}^{(1)}$ and $S_{zz}^{(1)}$ for a R–E fluid differ from those for the other two fluids.

4. SPHERICAL POLAR CO-ORDINATES

In this section we shall compare the steady secondary flows of the three fluids mentioned in § 1 using spherical polar co-ordinates.

Primary flow.—Taking the axis of rotation as the $\theta = 0$ axis, the primary flow is defined by

$$u_0 = v_0 = 0, \quad w_0 = w_0(r, \theta). \tag{4.1}$$

Here the non-vanishing stress components are

$$S_{r\phi}^{(0)} = r\frac{\partial}{\partial r}\left(\frac{w_0}{r}\right), \quad S_{\theta\phi}^{(0)} = \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{w_0}{\sin\theta}\right), \tag{4.2}$$

and w_0 is determined by

$$\frac{1}{r^3}\frac{\partial}{\partial r}\left\{r^4\frac{\partial}{\partial r}\left(\frac{w_0}{r}\right)\right\} + \frac{1}{r\sin^2\theta}\frac{\partial}{\partial\theta}\left\{\frac{\sin^3\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{w_0}{\sin\theta}\right)\right\} = 0, \tag{4.3}$$

with no slip condition at a solid boundary and the regularity of the solution at infinity if the fluid extends to infinity.

Secondary flow.—We record in Table II the non-zero stress components in the secondary flow.

TABLE II

	R–E	O	W
$S_{rr}^{(1)}$	$2u_r + (2K+S)[S_{r\phi}^{(0)}]^2$	$2u_r$	$2u_r$
$S_{\theta\theta}^{(1)}$	$\frac{2}{r}(u+v_\theta) + (2K+S)[S_{\theta\phi}^{(0)}]^2$	$\frac{2}{r}(u+v_\theta)$	$\frac{2}{r}(u+v_\theta)$
$S_{\phi\phi}^{(1)}$	$\frac{2}{r}(u+v\cot\theta) + S\,\bar{S}^{(0)}$	$\frac{2}{r}(u+v\cot\theta) + 2a\bar{S}^{(0)}$	$\frac{2}{r}(u+v\cot\theta) + 2a\bar{S}^{(0)}$
$S_{r\theta}^{(1)}$	$v_r + \frac{1}{r}u_\theta - \frac{v}{r} + (2K+S)\,S_{r\phi}^{(0)}\,S_{\theta\phi}^{(0)}$	$v_r + \frac{1}{r}u_\theta - \frac{v}{r}$	$v_r + \frac{1}{r}u_\theta - \frac{v}{r}$

where

$$\bar{S}^{(0)} = [S_{r\phi}^{(0)}]^2 + [S_{\theta\phi}^{(0)}]^2. \tag{4.4}$$

Here the stream function ψ defined by

$$u = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \tag{4.5}$$

satisfies the following equation

$$\begin{aligned} L^2[\psi] = R\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r}w_0^2\right) - \frac{\partial}{\partial r}(w_0^2\cot\theta)\right]\sin\theta \\ - 2a\sin\theta\left\{\frac{\partial}{\partial\theta}\left(\frac{1}{r}\bar{S}^{(0)}\right) - \frac{\partial}{\partial r}(\cot\theta\,\bar{S}^{(0)})\right\}, \end{aligned} \tag{4.6}$$

where

$$L \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{4.7}$$

We can specify the boundary conditions on the solid surface or surfaces as well as at infinity just as we have done in § 3.

We note the following important points:

(i) The secondary flow lies in the meridian planes and hence is orthogonal to the primary flow.

(ii) The effect of non-Newtonicity on ψ is again collected in the parameter a as in § 3. Therefore, all the three fluids have identical secondary flows for a given value of a.

(iii) The remarks (iii) and (iv) of § 3 for the secondary stress components hold good here also.

(iv) In the case of the rotational problems associated with one or two concentric spheres,[5] $w_0 = r\sin\theta w_0(r)$ so that $S_{\theta\phi}^{(0)} = 0$. Consequently all the secondary stress components for the three fluids are the same except that the normal stress components $S_{rr}^{(1)}$ and $S_{\phi\phi}^{(1)}$ for a R–E fluid differ from those for the other two fluids.

(v) In the case of the rotational problems associated with the cone-cone or cone-plate geometries,[6] $w_0 = r\sin\theta w_0(\theta)$, so that $S_{r\phi}^{(0)} = 0$. Consequently all the secondary stress components for these three fluids are the same except that the normal stress components $S_{\theta\theta}^{(1)}$ and $S_{\phi\phi}^{(1)}$ for a R–E fluid are different from those for the other two fluids.

5. Conclusion

Since the aim of the present note is just to compare, in a general manner, the physical characteristics of the non-Newtonian fluids defined by the constitutive equations given by Rivlin and Ericksen, Oldroyd, and Walters, we have not included the discussion of the phenomenon of separation and reversal of secondary flows, which is always present for appropriate values of a/R. Instead, we have referred to the relevant original investigations in the text.

It appears that with the help of the flows that we have considered in this note, we shall never be able to decide whether a given real fluid conforms to the constitutive equation (*a*) or (*b*) or (*c*). However, there is a possibility of deciding whether a given real fluid conforms to the constitutive equations given by Rivlin and Ericksen or the other two by studying the secondary stress components.

Finally, we can separate out the effects of the cross-viscosity and visco-elasticity for a R–E fluid by studying the phenomenon of separation and reversal of the secondary flow and simultaneously measuring the normal stress components.

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