COSMOLOGICAL APPLICATIONS OF MÖLLER'S TETRAD FORMULATION OF THE ENERGY-MOMENTUM COMPLEX

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ABSTRACT

A satisfactory energy-momentum complex has been given by Møller in terms of suitable tetrads introduced at every point of the space-time. A brief summary of Møller's work is given. Under suitable restrictions, Møller's theory could be extended so as to be applicable to energy problems in relativistic cosmology. Energy of various relativistic cosmological models is computed. It is found that the total energy of the flat universe or the de Sitter universe is zero but that of the Einstein universe of radius 2R₀ will be R₀. The energy distribution for a mass particle embedded in the homogeneous Einstein universe is also considered. The energy considerations lead to a new interpretation of the event horizon according to which the internal energy of a star, of radius equal to its Schwarzschild radius, is zero. This interpretation leads us to conclude that the Schwarzschild radius of a star embedded in the Einstein universe increases by about 4.8 per cent over its value when the background is considered to be empty.

1. INTRODUCTION

Møller (1961) has given a satisfactory solution of the problem of energy distribution in the framework of a tetrad formalism of the general relativity theory. Using Møller's theory it has been found (Shah, 1963) that the gravitational field outside a star embedded in an otherwise empty space does possess gravitational energy. For the Schwarzschild exterior field in spherically symmetric isotropic co-ordinates given by the line-element

\[
- ds^2 = \left(1 + \frac{m}{2r}\right)^4 [dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2] - \frac{(1 - \frac{m}{2r})^2}{\left(1 + \frac{m}{2r}\right)} dt^2.
\]

(1.1)
the total energy outside the star of radius \( r = r_1 \) and mass \( m \) is given by \( m^2/2r_1 \).

In order that the star behaves like a particle of mass \( m \) at large distances, it follows that the energy within the star must be \((m - m^2/2r_1)\). Using the internal solution for a static spherically symmetric material distribution in isotropic co-ordinates given by Narlikar and Vaidya (1942), we have verified that Møller's theory does give the same energy \((m - m^2/2r_1)\) within the boundary \( r = r_1 \) of the star. Hence for a star of mass \( m \) and radius \( r_1 = m/2 \) in these co-ordinates, the interior energy within its boundary becomes zero.

In terms of the Schwarzschild co-ordinates used in the line-element

\[
- ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} d\tilde{\tau}^2 + \tilde{\tau}^2 (d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2) - \left(1 - \frac{2m}{r}\right) dt^2,
\]

the boundary \( r_1 = m/2 \) is equivalent to \( \tilde{r}_1 = 2m \) for which \( g_{\tilde{4}\tilde{4}} \) becomes zero in (1.2) leading to the event horizon. Thus the energy consideration of spherical distributions in general relativity leads to a new interpretation for the event horizon that it is the boundary of the star of a given mass for which its internal energy vanishes. It will be now interesting to see whether this new interpretation leads to find any influence of the cosmological background of an individual star on its Schwarzschild radius.

The present paper deals with the cosmological applications of Møller's tetrad formulation of the problem of energy distribution in general relativity. A short summary of Møller's theory is given in Section 2. In Section 3, the total energy for various cosmological models is computed using Møller's energy-momentum complex. Section 4 discusses the energy problem for a star embedded in a cosmological background.

2. SUMMARY OF MøLLElL'S THEORY

In order to overcome the difficulties confronting Einstein's energy-momentum complex and the one proposed earlier by himself (Møller, 1958), Møller (1961) developed the theory for energy distribution in general relativity in the framework of a tetrad formalism. He obtained an energy-momentum complex \( \tau^i_k \) such that

\[
\tau^i_k = u^i_k l^l, l
\]

where the superpotential \( u^i_k l^l \) is

\[
u^i_k l = - u^i_k l = \frac{h}{8\pi} \left[ - h^k_h^l l^l + \left( \delta^i_k h^l - \delta^i_k h^k \right) h^s_s \right].
\]
Here \( h^i = h^i(x^i) \) are the sixteen components of a tetrad of orthogonal unit 4-vectors introduced at every point of the 4-space. \( \tau^i_k \) satisfies the conservation law \( \tau^k_{i,k} = 0 \).

Møller considered an arbitrary spherically symmetric static system in isotropic co-ordinates whose metric tensor is of the form

\[
g^{ik} = g_{ii} \delta_{ik}, \quad g^{ik} = \frac{\delta^{ik}}{g_{ii}},
\]

\[
g_{ii} = \{a(r), \ a(r), \ a(r), \ -b(r)\} \tag{2.3}
\]

where \( a(r) \) and \( b(r) \) are arbitrary functions of \( r \) only, \( x^4 = t \) and \( r^2 = (x^2)^2 + (x^3)^2 \) (The parenthesis after some index indicates that no summation over that index should be taken.) For this case the natural choice of the tetrads will be

\[
\hat{h}^i = \epsilon_a \frac{\delta_a^i}{\sqrt{|g_{ss}|}}, \quad \hat{h}^i = \sqrt{|g_{ss}|} \delta_{ai},
\]

\[
\hat{h}^i = \frac{\delta_a^i}{\sqrt{|g_{ss}|}}, \quad \hat{h}^i = \epsilon_a \sqrt{|g_{ss}|} \delta_{ai} \tag{2.4}
\]

where

\[\epsilon_a = \{1, 1, 1, -1\}.
\]

In order to get a complex \( \tau^i_k \) which is a unique function of the co-ordinates for a given physical system, Møller has supplemented a set of equations which single out the preferred tetrads to be used in the expression for \( \tau^i_k \). These equations are

\[
y_{ik}^1; i + y_{ik}^1 \Phi^1 = 0 \tag{2.5}
\]

where

\[
y_{ik} = -y_{ki} = \hat{h}^i h^k_{;i};
\]

and

\[
\Phi_k = y_{ik}^1 = -y_{ki}^1 = \hat{h}^i h^k_{;i}.
\]
3. Energy of Various Cosmological Models

In order to apply Møller's theory to static as well as non-static cosmological systems, we take the metric of the form

$$-ds^a = a(r, t) \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] - b(r, t) dt^2$$

with $x^a = t, r^a = (x^1)^2 + (x^2)^2 + (x^3)^2$ while $a(r, t)$ and $b(r, t)$ are arbitrary functions of $r$ and $t$. For the metric (3.1), we can still choose the tetrads in the form (2.4). Møller's uniqueness conditions (2.5) can then be exactly satisfied if

$$\frac{\partial}{\partial x^a} \frac{\partial}{\partial x^4} \log \left(\frac{b}{a}\right) = 0, \quad a = 1, 2, 3.$$  

All relativistic static as well as non-static cosmological models in spherically symmetric isotropic co-ordinates will satisfy conditions (3.2). We can now use (2.4), (2.2) and (2.1) to determine the energy of various cosmological models. Since $\tau^k_{,k} = 0$, the energy within a 3-space $\Sigma$ can be defined as

$$H = - \int_\Sigma \tau^a_{\,a} dx^1 dx^2 dx^3.$$  

For Minkowski's flat universe, as well as for de Sitter's empty universe, the energy density $\tau^a_{\,a}$ and hence the total energy $H$ turns out to be zero. In case of the Einstein static universe having the metric

$$-ds^a = \frac{1}{\left(1 + \frac{r^2}{4R_0^2}\right)^2} \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] - dt^2$$

where $2R_0$ is the constant radius and $r^a = (x^1)^2 + (x^2)^2 + (x^3)^2$, we get

$$\tau^a_{\,a} = - \frac{1}{8\pi R_0^2 \left(1 + \frac{r^2}{4R_0^2}\right)^2} \left[3 - \frac{r^2}{4R_0^2}\right],$$

$$H = R_0.$$  

For the non-static homogeneous universe given by the Robertson's metric

$$-ds^a = \frac{e^{\theta(t)}}{\left(1 + \frac{r^2}{4R_0^2}\right)^2} \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] - dt^2,$$

the total energy comes out to be $R_0 e^{\theta/2}$.

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* The results discussed in this section formed part of the thesis presented by the author for M.S. degree (1963) at the University of Maryland (U.S.A.).
4. A Mass Particle Embedded in a Static Einstein Universe

The exterior field of a mass particle embedded in a static Einstein universe is given by the metric (McVittie, 1933),

\[- ds^2 = \frac{1 + \frac{m}{2r} \left(1 + \frac{r^2}{4R_0^2}\right)^{\frac{3}{2}}}{\left(1 + \frac{r^2}{4R_0^2}\right)^{\frac{3}{2}}} \left[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\right]\]

\[- - \left(\frac{1 - \frac{m}{2r} \left(1 + \frac{r^2}{4R_0^2}\right)^{\frac{3}{2}}}{\left(1 + \frac{m}{2r} \left(1 + \frac{r^2}{4R_0^2}\right)^{\frac{3}{2}}\right)^{\frac{3}{2}}} \right)^2 dt^2.\]

For this system, (2.4), (2.2) and (2.1) give

\[- \tau_4^4 = \frac{1}{8\pi R_0^2} \left(1 + \frac{r^2}{4R_0^2}\right)^{\frac{3}{2}} \left\{3 - \frac{r^2}{4R_0^2}\right\} + \frac{m^2}{8\pi r^4} - \frac{3}{16 \pi r R_0^3} \left(1 + \frac{r^2}{4R_0^2}\right)^{-5/2}.\]

Using this in (3.3) with spherical polar co-ordinates we get the following expression for the total energy $H_0$ of the exterior field:

\[- H_0 = \int_0^{2\pi} \int_0^{\pi} \int_{r_1}^{R_0} \left[\frac{1}{8\pi R_0^3} \left(1 + \frac{r^2}{4R_0^2}\right)^{\frac{3}{2}} \left\{3 - \frac{r^2}{4R_0^2}\right\} + \frac{m^2}{8\pi r^4} - \frac{3}{16 \pi r R_0^3} \left(1 + \frac{r^2}{4R_0^2}\right)^{-5/2}\right] r^2 \sin \theta dr d\theta d\phi.\]

Integrating this we get

\[- H_0 = R_0 - \frac{r_1^3}{2R_0^2} \left(1 + \frac{r_1^2}{4R_0^2}\right)^{-3} + \frac{m^3}{2r_1} \left(1 - \frac{r_1}{2R_0}\right) + (2)^{-3/2} m - m \left(1 + \frac{r_1^2}{4R_0^2}\right)^{-3/2}.\]

Here $r = r_1$ is the outer boundary of the particle. Assuming that the total energy of the particle and the background universe is the same as that of the Einstein universe, i.e., $R_0$, we get the energy $H_i$ within the particle of radius $r_1$, as given by
Neglecting terms of the order of \( r_1/R_0 \), one gets,

\[
H_1 = \left[1 - \left(2 - \frac{r_1^2}{4R_0^2}\right)^{-3/2} \right] - \frac{m^2}{2r_1} \left(1 - \frac{r_1}{2R_0}\right)
\]

\[+ \frac{r_1^3}{2R_0^3} \left(1 + \frac{r_1^2}{4R_0^2}\right)^{-2}.
\]

If we now use the energy interpretation of the Schwarzschild radius and put \( H_1 = 0 \), we get the Schwarzschild radius of the particle in the Einstein universe which will be \( r_1 = 0.7734 \text{ m} \). This, when couched in the Schwarzschild co-ordinates (1.2), becomes \( \tilde{r}_1 = 2.096 \text{ m} \). Since for the same particle embedded in an empty background, the Schwarzschild radius is \( r^* = 2m \), we conclude that the presence of the cosmological background increases the radius of the event horizon by about 4.8 per cent.

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