PULSATING MAGNETOHYDRODYNAMIC FLOW IN AN ANNULAR CHANNEL

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Abstract

Pulsating laminar flow of a viscous incompressible electrically conducting fluid in an annular channel between two infinitely long circular cylinders under a radial impressed magnetic field is considered. The solutions of magnetohydrodynamic equations have been obtained on the assumption that the space between two cylinders is small compared to their mean radius. The solutions were also obtained on the assumption of small magnetic Reynolds number with special consideration of those for low and high frequencies.

1. Introduction

In his recent paper, Narasimhan[1] (1964) obtained the solution of magnetohydrodynamic (hereafter called MHD) equations under the approximations $(\nu/\nu_m)^{\frac{1}{2}} \ll 1$ and $\omega_0 \sim (\nu/\nu_m)^{\frac{1}{2}}$, where ν is the viscosity, ν_m the magnetic viscosity and ω_0 the frequency. Similar types of problems have been investigated by Rudraiah and Blackwell (1963).[2] To obtain his solutions, Narasimhan (1964)[1] assumed that the cylinders are infinitely long insulators and at the same time he used the boundary condition $(dH_z/dr)|_{r=b} = 0$. This boundary condition is true only when the walls of the cylinders are perfect conductors. To derive his non-steady state solution, Narasimhan assumed that $(\nu/\nu_m)^{\frac{1}{2}} = (R_m/R)^{\frac{1}{2}} \ll 1$ and $\omega_0 \sim (\nu/\nu_m)^{\frac{1}{2}}$. In this paper, we try to demonstrate that Narasimhan's solution can also be obtained on the approximation of the magnetic Reynolds number $R_m = \omega_0 a^2/\nu_m \ll 1$. The solutions were also obtained under the narrow-gap approximation which will be useful in some experimental work, [*e.g.*, *see* Chandrasekhar (1961)[3]].

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2. BASIC EQUATIONS

We assume that the cylinders, of radii a and b ($b > a$), are infinitely long perfect conductors. The equations governing the motion of an electrically conducting incompressible viscous fluid in the presence of electric and magnetic fields, under the usual approximations (Cowling, 1957)[4] are (in a rationalized absolute unit system)

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right] = \mu\vec{J} \times \vec{H} - \nabla\pi - \rho\nu(\nabla \times \nabla \times \vec{V}) \tag{1}$$

$$\nabla\cdot\vec{V} = 0 \tag{2}$$

$$\frac{\partial \vec{H}}{\partial t} = \nabla \times (\vec{V} \times \vec{H}) - \nu_m(\nabla \times \nabla \times \vec{H}) \tag{3}$$

$$\nabla\cdot\vec{H} = 0 \tag{4}$$

where

$\vec{V}$, $\vec{H}$, π, t are the velocity, magnetic field, pressure and time, ρ, μ, σ, ν, ν_m are the density, magnetic permeability, electrical conductivity, kinematic viscosity and magnetic viscosity ($= 1/\mu\sigma$).

Let (r, θ, Z) be cylindrical co-ordinates referred to the common axis of the cylinders as Z-axis.

The problem is assumed two-dimensional and axisymmetric with velocity entirely axial and the fluid is driven by a (axial) pressure gradient where magnitude is sinusoidal in time. The applied magnetic field is in the radial direction and the resulting induced magnetic field is then entirely axial (Kapur and Jain, 1960).[5] Then the special condition of the problem requires:

$$\nabla\pi = -\mathrm{P}e^{i\omega_0 t}\hat{Z} + \nabla\pi_i. \tag{5}$$

(P, ω_0 real and $\mathrm{P} > 0$),

$$\vec{V} = \mathrm{W}(r)\, e^{i\omega_0 t}\hat{Z} \tag{6}$$

$$\vec{H} = \frac{\mathrm{A}}{r}\hat{r} + \vec{H}_i \tag{7}$$

where $\mathrm{H}_r = \mathrm{A}/r$ ($\mathrm{A} > 0$) is an applied magnetic field (Globe, 1959), Substituting (5) and (6) into (1) and (3) using (2) and (4) we find that the induced pressure gradient $\nabla \pi_i$ is entirely in the radial direction and $\vec{H}_i$ is entirely in the axial direction (Kapur and Jain, 1960). In fact if

$$\pi_i = p e^{i\bar{\omega}_0 t}\,\hat{r}, \quad \vec{H}_i = H_Z e^{i\bar{\omega}_0 t}\,\hat{Z} \tag{8}$$

we obtain

$$\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial r} = -\frac{\mu H_Z}{\rho}\frac{dH_Z}{dr} \tag{9}$$

$$i\omega_0 W - \frac{\mu A}{\rho r}\frac{dH_Z}{dr} = \frac{P}{\rho} + \nu\left(\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr}\right) \tag{10}$$

$$i\omega_0 H_Z - \frac{A}{r}\frac{dW}{dr} = \nu_m\left(\frac{d^2H_Z}{dr^2} + \frac{1}{r}\frac{dH_Z}{dr}\right). \tag{11}$$

Although exact solutions of equations (9) to (11) are possible, it is believed that, in many practical applications, approximate limiting solutions of differing physical significances are likely to be more useful and such solutions are considered in this paper.

3. The Narrow-Gap Solution

We assume that $d/a \ll 1$, where $d = b - a$ is the gap width. In this approximation equations (9) to (11) become

$$\frac{1}{\rho}\frac{\partial p}{\partial \Sigma} = -\frac{\mu}{\rho} H_Z \frac{dH_Z}{d\Sigma} \tag{12}$$

$$i\omega_0 W - \frac{\mu A}{\rho a d}\frac{dH_Z}{d\Sigma} = \frac{P}{\rho} + \frac{\nu}{d^2}\left(\frac{d^2W}{d\Sigma^2}\right) \tag{13}$$

$$i\omega_0 H_Z - \frac{A}{ad}\frac{dW}{d\Sigma} = \frac{\nu_m}{d^2}\left(\frac{d^2H_Z}{d\Sigma^2}\right) \tag{14}$$

where

$$\Sigma = \frac{r-a}{d}, \quad 0 \leqslant \Sigma \leqslant 1$$

we then define

$$X(\Sigma) = \int_0^{\Sigma} H_Z(\lambda)\, d\lambda. \tag{15}$$

Substituting in equation (14) from equation (15) one then obtains

$$W = \frac{ad}{A}\left[i\omega_0 X - \frac{\nu_m}{d^2}\frac{d^2X}{d\Sigma^2}\right]. \tag{16}$$

From equation (13), using equation (16), we get

$$\frac{d^4X}{d\Sigma^4} - A_1\frac{d^2X}{d\Sigma^2} - A_2X = A_3 \tag{17}$$

where

$$A_1 = i(R + R_m) + M^2,\ A_2 = RR_m,\ A_3 = \frac{APd^3}{\rho a^2 \nu \nu_m}.$$

$R = \frac{\omega_0 d^2}{\nu}$ is the Reynolds number,

$R_m = \frac{\omega_0 d^2}{\nu_m}$ is the magnetic Reynolds number,

$M = \left[\frac{MA^2 d^2}{\rho a^2 \nu \nu_m}\right]^{\frac{1}{2}}$ is the Hartmann number.

To solve equation (17), one needs four boundary conditions on $X(\Sigma)$. They are as follows:

The first two are the no-slip boundary conditions on velocity and they are

$$W(a, t) = W(b, t) = 0. \tag{18}$$

The remaining two boundary conditions depend on the magnetic field. The current density $\vec{J}$ has only a θ-component, so that the current in the annular channel is analogous to that in an infinite solenoid and may be assumed to produce no field for $r > b$ (Globe, 1959).[6] Thus continuity of tangential component of magnetic field requires that

$$H_z(b, t) = 0. \tag{19}$$

Since the walls of the cylinders are perfect conductors, the tangential component of electric field has to be continuous and hence zero at the walls. Therefore

$$E(a, t) = E(b, t) = 0. \tag{20}$$

Since

$$j_\theta = \sigma\,[\,E_\theta + \mu(\vec{V} \times \vec{H})_\theta] = -\frac{dH_z}{dr}$$

then from (18) to (20) it follows that

$$\left.\frac{dH_z}{dr}\right|_{r=b} = 0. \tag{21}$$

Hence the required boundary conditions on $X(\Sigma)$ are:

$$X(0) = X(1) = 0$$

$$\left.\frac{dX}{d\Sigma}\right|_{\Sigma=1} = \left.\frac{d^2X}{d\Sigma^2}\right|_{\Sigma=1} = 0. \tag{22}$$

The solution of (17) subject to the boundary conditions (22) is

$$X(\Sigma) = d_1 \cosh \lambda_1\Sigma + d_2 \sinh \lambda_1\Sigma + d_3 \cosh \lambda_2\Sigma + d_4 \sinh \lambda_2\Sigma - \frac{A_3}{A_2} \tag{23}$$

where

$$d_1 = \frac{\lambda_2 A_3}{A_2 D}\,[\lambda_1\lambda_2 \sinh \lambda_2 \cosh \lambda_1 - \lambda_1^2 \cosh \lambda_2 \sinh \lambda_1 - (\lambda_2^2 - \lambda_1^2) \sinh \lambda_1]$$

$$d_2 = \frac{\lambda_2 A_3}{A_2 D}\,[(\lambda_2^2 - \lambda_1^2) \cosh \lambda_1 - \lambda_1\lambda_2 \sinh \lambda_2 \sinh \lambda_1 + \lambda_1^2 \cosh \lambda_1 \cosh \lambda_2 - \lambda_2^2]$$

$$d_3 = \frac{\lambda_1 A_3}{A_2 D}\,[(\lambda_2^2 - \lambda_1^2) \sinh \lambda_2 + \lambda_1\lambda_2 \sinh \lambda_1 \cosh \lambda_2 - \lambda_2^2 \cosh \lambda_1 \sinh \lambda_2]$$

$$d_4 = \frac{\lambda_1 A_3}{A_2 D}\,[\lambda_2^2 \cosh \lambda_2 \cosh \lambda_1 - \lambda_1\lambda_2 \sinh \lambda_1 \sinh \lambda_2 - (\lambda_2^2 - \lambda_1^2) \cosh \lambda_2 - \lambda_1^2]$$

$$D = (\lambda_2^2 - \lambda_1^2)(\lambda_1 \sinh \lambda_2 - \lambda_2 \sinh \lambda_1).$$

From (16) and (15) using (23) the expressions for velocity and magnetic field are:

$$\mathrm{W}(\Sigma) = \Big[a\,(d_1 \cosh \lambda_1\Sigma + d_2 \sinh \lambda_1\Sigma) + a_2\,(d_3 \cosh \lambda_2\Sigma$$

$$+ d_4 \sinh \lambda_2\Sigma) - i\,\frac{\omega_0 a d \mathrm{A}_3}{\mathrm{A}\mathrm{A}_2}\Big] \tag{24}$$

$$\mathrm{H}(\Sigma) = \lambda_1 d_1 \sinh \lambda_1\Sigma + \lambda_1 d_2 \cosh \lambda_1\Sigma + \lambda_2 d_3 \cosh \lambda_2\Sigma$$

$$+ \lambda_2 d_4 \sinh \lambda_2\Sigma \tag{25}$$

where

$$a_1 = \frac{ad}{\mathrm{A}}\left(i\omega_0 - \frac{\nu_m \lambda_1^2}{d^2}\right), \quad a_2 = \frac{ad}{\mathrm{A}}\left(i\omega_0 - \frac{\nu_m \lambda_2^2}{d_2}\right)$$

$$2\lambda_1 = \xi_1 + i\xi_2, \quad 2\lambda_2 = \eta_1 + i\eta_2$$

$$\xi_1, \xi_2 = \frac{1}{2^{\frac{1}{2}}}\left[\{((\mathrm{R}^{\frac{1}{2}} + \mathrm{R}_m^{\frac{1}{2}})^4 + \mathrm{M}^4)^{\frac{1}{2}} \pm \mathrm{M}^2\}^{\frac{1}{2}} + \{((\mathrm{R}^{\frac{1}{2}} - \mathrm{R}_m^{\frac{1}{2}})^4\right.$$

$$\left. + \mathrm{M}^4)^{\frac{1}{2}} \pm \mathrm{M}^2\}^{\frac{1}{2}}\right]$$

$$\eta_1, \eta_2 = \frac{1}{2^{\frac{1}{2}}}\left[\{((\mathrm{R}^{\frac{1}{2}} + \mathrm{R}_m^{\frac{1}{2}})^4 + \mathrm{M}^4)^{\frac{1}{2}} \pm \mathrm{M}^2\}^{\frac{1}{2}} - \{((\mathrm{R}^{\frac{1}{2}} - \mathrm{R}_m^{\frac{1}{2}})^4\right.$$

$$\left. + \mathrm{M}^4)^{\frac{1}{2}} \pm \mathrm{M}^2\}^{\frac{1}{2}}\right].$$

Expressions (24) and (25), in general, are very complicated. These expressions, however, will reduce to a simpler form with the use of the conditions appropriate to laboratory or cosmical applications of MHD [*e.g., see* Steketee (1959),[7] Ludford (1959),[8] Axford (1960)[9]]. These conditions are $\mathrm{N} \ll 1$, $\mathrm{N} \gg 1$ and $\mathrm{N} = 1$, where $\mathrm{N} = \mathrm{R}_m/\mathrm{R}$ is the magnetic Prandtl number.

Case 1 ($\mathrm{N} \ll 1$):

Under laboratory conditions it is usually found that $\mathrm{N} = \nu/\nu_m \ll 1$ ($\nu/\nu_m = 10^{-6}/6$ for mercury under laboratory conditions). In this approximation the expressions for λ_1 and λ_2 are:

$$\lambda_1 = (\mathrm{M}^2 + i\mathrm{R})^{\frac{1}{2}} = f_1 + if_2$$

$$\lambda_2 = \frac{i\,(\mathrm{R}\mathrm{R}_m)^{\frac{1}{2}}}{(\mathrm{M}^2 + i\mathrm{R})^{\frac{1}{2}}} = g_1 + ig_2$$

where

$$f_1, f_2 = \frac{1}{2^{\frac{1}{2}}}\left[(R^2 + M^4)^{\frac{1}{2}} \pm M^2\right]^{\frac{1}{2}}$$

$$g_1 = \left(\frac{RR_m}{R^2 + M^4}\right)^{\frac{1}{2}} f_2, \quad g_2 = \left(\frac{RR_m}{R^2 + M^4}\right)^{\frac{1}{2}} f_1.$$

Using these expressions for λ_1 and λ_2 the expressions for velocity and magnetic field can then easily be obtained from equations (24) and (25) respectively.

Case 2 ($N \gg 1$):

Under cosmical conditions it is usually found that $N \gg 1$ ($\nu/\nu_m \approx 10^{13}$ for corona). In this case, the expressions for λ_1 and λ_2 are:

$$\lambda_1 = (M^2 + iR_m)^{\frac{1}{2}}, \quad \lambda_2 = \frac{i(RR_m)^{\frac{1}{2}}}{(M^2 + iR_m)^{\frac{1}{2}}}.$$

Case 3 ($N = 1$):

In somewhat artificial case when $N = 1$, *i.e.*, $\nu = \nu_m$, the expressions for λ_1 and λ_2 will be simplified and they are given by

$$\lambda_1, \lambda_2 = \frac{1}{2^{\frac{1}{2}}}\left[\left\{\left(R_m^2 + \frac{M^4}{16}\right)^{\frac{1}{2}} + \frac{M^2}{4}\right\}^{\frac{1}{2}} + i\left\{\left(R_m^2 + \frac{M^4}{16}\right)^{\frac{1}{2}} - \frac{M^2}{4}\right\}^{\frac{1}{2}}\right] \pm \frac{M}{2}.$$

4. Solutions with Small Magnetic Reynolds Number and Without the Narrow-Gap Approximation

Equations (24) and (25) will reduce to simpler form on the approximation of a small magnetic Reynolds number. In many practical applications it is usually found that the magnetic Reynolds number $R_m \ll 1$. Hence in this section we shall assume that $R_m \ll 1$.

Introducing the non-dimensional quantities

$$\alpha = \frac{r}{a}, \quad V = \frac{W}{a\omega_0}, \quad H = \frac{aH_z}{A}, \quad p_1 = \frac{p}{\rho a^2 \omega_0^2}$$

into equations (9) to (11) and using $R_m = \omega_0 a^2/\nu_m \ll 1$, one then obtains

$$\frac{\partial p_1}{\partial a} = - SH \frac{dH}{da} \tag{26}$$

$$\frac{d^2V}{da^2} + \frac{1}{a}\frac{dV}{da} + \frac{SR}{a}\frac{dH}{da} - iRV = - P_1 \tag{27}$$

$$\frac{d^2H}{da^2} + \frac{1}{a}\frac{dH}{da} = - \frac{R_m}{a}\frac{dV}{da} \tag{28}$$

where

$$P_1 = \frac{PR}{a\omega_0{}^2}, \quad S = \frac{\mu A^2}{\rho a^4 \omega_0{}^2}, \quad R = \frac{\omega_0 a^2}{\nu}, \quad R_m = \frac{\omega_0 a^2}{\nu_m}.$$

The corresponding boundary conditions, from equations (18) to (21), are

$$V(1) = V(\gamma) = 0$$

$$H(\gamma) = \frac{dH}{da}\bigg|_{a=\gamma} = 0 \tag{29}$$

where $\gamma = b/a$.

Equation (28) may then be integrated yielding

$$\frac{dH}{da} = - \frac{R_m}{a} V \tag{30}$$

where boundary condition (29) has removed the constant of integration. Equation (30) may be used to eliminate H from equation (27) and then we obtain

$$\frac{d^2V}{da^2} + \frac{1}{a}\frac{dV}{da} + \left(\beta^2 - \frac{M^2}{a^2}\right)V = - P_1 \tag{31}$$

where $\beta^2 = - iR$, $M = (SRR_m)^{\frac{1}{2}}$ is the Hartmann number. Equation (31) is a non-homogeneous modified Bessel equation with complex argument, which is exactly that found by Narasimhan (1964)[1] with two approximations $(R_m/R)^{\frac{1}{2}} \ll 1$ and $\omega_0 \sim (R_m/R)^{\frac{1}{2}}$ whereas equation (31) is derived with an approximation $R_m \ll 1$. One may express the solution of (31) in terms of Kelvin and Lommel functions (*e.g.*, *see* Narasimhan, 1964). However, for convenience, we can express the solution of (31) satisfying the boundary conditions (29) in the form

$$V(a) = [BJ_M(\beta a) + CY_M(\beta a)] \tag{31}$$

where

$$B = -\frac{\pi}{2}\left[\frac{Y_M(\beta)\,I}{\triangle} + P_1\int_1^a Y_M(\beta\lambda)\,\lambda d\lambda\right]$$

$$C = \frac{\pi}{2}\left[\frac{J_M(\beta)\,I}{\triangle} + P_1\int_1^a J_M(\beta\lambda)\,\lambda d\lambda\right]$$

$$\triangle = J_M(\beta)\,Y_M(\gamma\beta) - J_M(\gamma\beta)\,Y_M(\beta),$$

$$I = P_1 J_M(\gamma\beta)\int_1^\gamma aY_M(\beta a)\,da - P_1 Y_M(\gamma\beta)\int_1^\gamma aJ_M(\beta a)\,da.$$

$J_m(x)$ and $Y_m(x)$ are the Bessel functions of the first and second kinds, of order M, respectively.

Using equation (31) and the boundary condition (29), the magnetic field H may be determined from equation (30).

In the two limiting cases of small and large $|\beta|$ equation (31) may be expressed approximately in terms of elementary functions.

Low Frequency Oscillations ($|\beta|\,b \ll 1$):

In this case, using the ascending series expansions for the Bessel functions, to order $|\beta b^2|$ the expression (31) becomes:

$$V_t(a,t) \approx -\frac{P_1 e^{i\omega_0 t}}{(M^2-4)}\left[\frac{(\gamma^{2+M}-1)\,a^M}{(\gamma^{2M}-1)} - \frac{\gamma^{2M}(\gamma^{2-M}-1)}{(\gamma^{2M}-1)\,a^M} - a^2\right] \tag{32}$$

where $V_t(a,t)$ is the time-dependent velocity obtained from (6) using (31). In real notation

$$V_t(a,t) \approx -\frac{P_1\cos\omega_0 t}{(M^2-4)}\left[\frac{(\gamma^{2+M}-1)\,a^M}{(\gamma^{2M}-1)} - \frac{\gamma^{2M}(\gamma^{2-M}-1)}{(\gamma^{2M}-1)\,a^M} - a^2\right]. \tag{33}$$

The expression for magnetic field, from (30) using (29) and (33), is

$$H_t(a,t) \approx \frac{MP_1\cos\omega_0 t}{(M^2-4)}\left[\frac{(\gamma^{2+M}-1)(\gamma^M-a^M)}{M(1-\gamma^{2M})} + \frac{\gamma^{2M}(\gamma^{2-M}-1)(a^{-M}-\gamma^{-M})}{M(\gamma^{2M}-1)} - \frac{1}{2}(a^2-\gamma^2)\right]. \tag{34}$$

It is apparent that the solutions given by (33) and (34) are true only when $M^2 \neq 4$. However, for $M^2 = 4$ the singular solutions exist and they are given by

$$V_t(a, t) \approx -\frac{P_1 \cos \omega_0 t}{4(\gamma^4 - 1)}\left[\gamma^4\left(\log \frac{a}{\gamma}\right)a^2 + \frac{\gamma^4 \log \gamma}{a^2} - a^2 \log a\right] \quad (35)$$

$$H_t(a, t) \approx \frac{MP_1 \cos \omega_0 t}{4(\gamma^4 - 1)}\left[\frac{\gamma^4 a^2 \log \frac{a}{\gamma}}{2} + \gamma^2 \log \gamma - \frac{\gamma^4 \log \gamma}{2a^2} - \frac{a^2}{2}\log a + \frac{(\gamma^4 - 1)(\gamma^2 - a^2)}{4}\right]. \quad (36)$$

In the limit $M \to 0$, equation (33) approaches the hydrodynamic limit

$$V_t(a, t) \approx -\frac{P_1 \cos \omega_0 t}{4}\left[a^2 - \frac{(\beta^2 - 1)\log a}{\log \beta} - 1\right]. \quad (37)$$

Large Oscillations ($|\beta a| \gg 1$):

When the oscillations are large, β assumes very large values. We know that for large values of Z (Hildebrand[10])

$$J_M(Z) \sim \sqrt{\frac{2}{\pi Z}} \cos\left(Z - \frac{M\pi}{2} - \frac{\pi}{4}\right)$$

$$Y_M(Z) \sim \sqrt{\frac{2}{\pi Z}} \sin\left(Z - \frac{M\pi}{2} - \frac{\pi}{4}\right).$$

When these values are substituted in equation (31) we get

$$V_t(a, t) \approx \frac{P_1 e^{i\omega_0 t}}{\beta a^{\frac{1}{2}}}\left[\frac{\sin \beta(a - 1)}{\sin \beta(\gamma - 1)} M(\lambda, \gamma) - M(\lambda, a)\right] \quad (38)$$

where

$$M(\lambda, x) = \int_1^X \sin \beta(\lambda - X)\frac{d\lambda}{\lambda^{\frac{1}{2}}}.$$

5. CONCLUSIONS

In order to study the effect of the magnetic field on the pulsatory MHD flow, the hydromagnetic solution (32) has been evaluated for M = 0, 3, 6, 12 and for a fixed $\omega_0 t$ where $M = 0$ is the hydrodynamic case. Typical behaviour is illustrated with the family of curves in Fig. 1 which refers to $d = 2$. It is seen that the magnetic field flattens the velocity profile, which suggests that the field tends to inhibit instability.

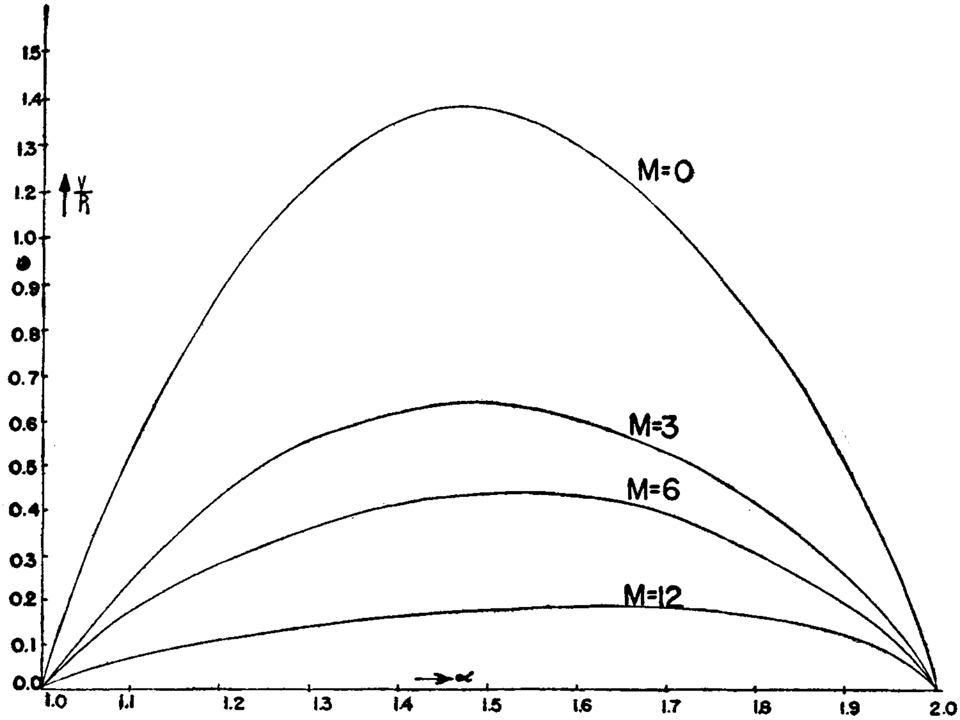

Fig. 1. Velocity profile of Magnetohydrodynamic flow in an Annular Channel.

The problem of pulsating MHD flow in an annular channel with porous walls is presented in another paper.

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6. References

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