BIRKHOFF'S RINGS IN THE RESTRICTED PROBLEM OF THREE BODIES IN A THREE-DIMENSIONAL CO-ORDINATE SYSTEM

By Ram Kishore Choudhry

(Department of Mathematics, L. S. College, Muzaffarpur, Bihar University, Bihar)

Received August 14, 1964

(Communicated by Dr. N. S. Nagendra Nath, F.A.Sc.)

ABSTRACT

In this paper, firstly the trajectory corresponding to the solution of the restricted problem of 3-bodies in a 3-dimensional co-ordinate system has been represented on a 3-dimensional torus and finally, it is seen that the points on the trajectory have one-to-one correspondence with the points on the rings which were studied by Birkhoff [1] for the two-dimensional problem. The special feature of the paper lies in the generalisation of Levi-Civita transformation for a 3-dimensional co-ordinate system to regularise the solutions.

1. INTRODUCTION

In the earlier paper [4] in Art. 4, the existence of two circular motions, one direct and another retrograde in the problem of two bodies for \( C \) (Jacobi's constant) \( > 3 \) has been shown. During the study of this existence it was found that there is a chance of collision. Later on analytical continuations of these circular motions have been studied. Also the parametric representation of these continuations in cylindrical co-ordinate system is found in the form [4, 6·4]:

\[
\begin{align*}
\rho' &= \rho \left( \sqrt{1 - \mu} + \mu \rho^2 f(\rho, \theta, \mu) \right) \\
\theta' &= \theta + \mu \rho^2 g(\rho, \theta, \mu) \\
z &= \rho^2 \mu h(\rho, \theta, \mu)
\end{align*}
\]

where \( f, g, h \) are analytic in \( \rho \) and \( \mu \) and periodic in \( \theta \) of period \( 2\pi \).

In the paper [2] it is seen that Birkhoff's rings play an important role for any qualitative study of the orbits in the plane restricted problem of three bodies. Aiming that it might be equally useful for three-dimensional case, I took up this problem.

252
253

Birkhoff's Rings in Three-Dimensional Co-ordinate System

So far as notations and references in this paper are concerned, the reader is expected to have knowledge of my paper [4]. Sometimes references to this paper have been made; for example, if some result is referred to [4, 1.2], then the symbol [4] stands for my earlier paper and (1.2) stands for the second result in the first section.

2. ANALYTICAL CONTINUATIONS OF THE CIRCULAR ORBITS

Though we have already found the parametric representation of the analytical continuation [4, 6.4], but it remains to show that it is quite different from Birkhoff's analytical continuation [2, page 719]. To examine this point it may be noted that if \( h(\rho, \theta, \mu) \) is identically zero, the problem reduces to Birkhoff’s case. So to distinguish our case from that of Birkhoff it is necessary to show that \( h \) is not identically zero, i.e., \( h_1 \) in the following representation given in the same article [4, Art. 6] is not identically zero:

\[
\begin{align*}
\xi &= (1 - \mu) \cos (1 - \mu)^{-1} \tau + \lambda^3 f_1 (\lambda, \mu, \tau) \\
\eta &= (1 - \mu) \sin (1 - \mu)^{-1} \tau + \lambda^3 g_1 (\lambda, \mu, \tau) \\
\zeta &= \lambda^3 h_1 (\lambda, \mu, \tau)
\end{align*}
\]

(1)

Now \( \xi \) is obtained by solving the equation [4, 5.3]

\[
\frac{d^2 \xi}{d\tau^2} = -\frac{(1 - \mu) \xi}{(\xi^2 + \eta^2 + \zeta^2)^{3/2}} - \frac{\mu \lambda^4 \xi}{((1 + \lambda^2 \xi)^2 + \lambda^4 \eta^2 + \lambda^4 \zeta^2)^{3/2}}.
\]

(2)

Putting the value of \( \xi, \eta \) and \( \zeta \) in

\[
(\xi^2 + \eta^2 + \zeta^2)^{3/2}
\]

and

\[
((1 + \lambda^2 \xi)^2 + \lambda^4 \eta^2 + \lambda^4 \zeta^2)^{3/2},
\]

we find that

\[
(\xi^2 + \eta^2 + \zeta^2)^{-3/2}
\]

\[
= \frac{1}{(1 - \mu)^3} \left[ 1 - 3 \frac{\lambda^2}{1 - \mu} \left( f_1 \cos (1 - \mu)^{-1} \tau + g_1 \sin (1 - \mu)^{-1} \tau \right) \right.
\]

\[
+ 0 (\lambda^4) \left. \right] + 0 (\lambda^4)
\]

\[
= 1 - 3 \lambda^2 (1 - \mu) \cos (1 - \mu)^{-1} \tau + 0 (\lambda^4)
\]

A3
\[
\begin{align*}
\left(1 - \mu\right) \zeta &= \frac{\lambda^2 h_1(\lambda, \mu, \tau)}{(1 - \mu)^2} + 0(\lambda^4) \\
\left(1 + \lambda^2 \xi \right)^2 + \lambda \eta^2 + \lambda^2 \xi \zeta \frac{1}{\sqrt{2}} &= \mu \lambda^8 \zeta \\
(1 + \lambda^2 \xi)^2 + \lambda \eta^2 + \lambda^2 \xi \zeta \frac{1}{\sqrt{2}} &= \mu \lambda^8 h_1(\lambda, \mu, \tau) + 0(\lambda^4)
\end{align*}
\]

\[
(2) \text{ reduces to}
\lambda^2 \frac{d^2 h_1}{d \tau^2} = -\lambda^2 \frac{h_1}{(1 - \mu)^2}
\tag{3}
\]

if terms containing powers of \(\lambda\) higher than the second be neglected.

As \(h_1(\lambda, \mu, \tau)\) is analytic in \(\lambda\) and \(\mu\), so it can be expanded in ascending powers of \(\lambda\), thus we can put
\[
h_1(\lambda, \mu, \tau) = a_0(\mu, \tau) + \lambda a_1(\mu, \tau) + \ldots
\]

As we are restricted only with the second powers of \(\lambda\), so \(h_1(\lambda, \mu, \tau) = a_0(\mu, \tau)\), for higher power in \(\lambda\) will give a term to \(\zeta\) with power of \(\lambda\) higher than the second. Thus from (3)
\[
\frac{d^2 a_0}{d \tau^2} = -\frac{a_0}{(1 - \mu)^2}
\]

\[
\therefore a_0(\mu, \tau) = A \cos(1 - \mu)^{-1} \tau + B \sin(1 - \mu)^{-1} \tau.
\]

Since initially for \(\tau = 0\), \(\zeta = 0\) and \(d\zeta/d\tau > 0\) which means that \(a_0(\mu, 0) = 0\) and \((da_0(\mu, \tau)/d\tau) > 0\).

\[
\therefore A = 0 \quad \text{and} \quad B = (1 - \mu) \left(\frac{da_0(\mu, \tau)}{d\tau}\right)_{\tau = 0} \neq 0.
\]

It shows that
\[
a_0(\mu, \tau) = (1 - \mu) \left(\frac{da_0(\mu, \tau)}{d\tau}\right)_{\tau = 0} \sin(1 - \mu)^{-1} \tau.
\]

Thus \(\zeta\) being equal to
\[
\lambda^2 (1 - \mu) \left(\frac{da_0(\mu, \tau)}{d\tau}\right)_{\tau = 0} \sin(1 - \mu)^{-1} \tau
\]

for all \(\tau\) is not identically zero. Hence it is shown that the present analytical continuation is quite different from that of Birkhoff.

3. Regularisation of the Solution [4, 6.4]

We have already seen in Art. 4 of [4] that under our consideration there is a chance of collision with the mass \(P_1\). Thus the solution becomes singular
in the neighbourhood of this moment. It has been shown by Sundman [8, pp. 90–102] that a transformation can always be chosen to remove a singularity caused by collision. With this aim, let us introduce the transformation of Levi-Civita [5] which can be easily generalised to 3-dimensions in the form:

\[
\begin{align*}
x - \mu &= p^2 - q^2, \\
y &= 2pq, \\
z &= z \\
dt &= 4r_1 \, d\tau, \\
r_1 &= \sqrt{(x - \mu)^2 + y^2 + z^2}
\end{align*}
\]

(4)

(It is understood that \( r \) introduced here is different from \( r \) in [4, 5.3]).

Let the differential equations [4, 2.1] be put in the form:

\[
\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \quad (i = 1, 2, 3)
\]

(5)

where

\[
\begin{align*}
x_1 &= x - \mu, \\
y_1 &= \dot{x} - \mu \\
x_2 &= y, \\
y_2 &= \dot{y} + x - \mu \\
x_3 &= z, \\
y_3 &= \dot{z}
\end{align*}
\]

\[
F + C = \frac{1}{2} [\dot{x}^2 + \dot{y}^2 + \dot{z}^2] - \Omega + \frac{C}{2} = 0,
\]

and

\[
\Omega = \frac{1}{2} (x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}.
\]

Let us put the transformation (4) in the form

\[
x_1 + ix_2 = (p + iq)^2, \quad (y_1 - i y_2) = \frac{\bar{\omega} - i\chi}{2(p + iq)}
\]

then we find that

\[
y_1 dx_1 + y_2 dx_2 + y_3 dx_3 = \bar{\omega} dp + \chi dq + y_3 dx_3.
\]

Thus it shows that the canonical character remains preserved and equations can be written as follows:

\[
\begin{align*}
\frac{dp}{dt} &= \frac{\partial F}{\partial \omega}, \\
\frac{dq}{dt} &= \frac{\partial F}{\partial \chi}, \\
\frac{dx_3}{dt} &= \frac{\partial F}{\partial x_3}
\end{align*}
\]

\[
\begin{align*}
\frac{d\bar{\omega}}{dt} &= -\frac{\partial F}{\partial p}, \\
\frac{d\chi}{dt} &= -\frac{\partial F}{\partial q}, \\
\frac{dy_3}{dt} &= -\frac{\partial F}{\partial x_3}
\end{align*}
\]

(6)
where
\[ x_1 = p^2 - q^2, \quad x_2 = 2pq, \quad x_3 = z \]
\[ \dot{x} = 2(p\dot{p} - q\dot{q}), \quad \dot{y} = 2(p\dot{q} + \dot{p}q), \quad \dot{z} = y_3. \]

Taking into consideration the further study of the problem, we shall prefer to express Jacobi’s integral in terms of \( p \) and \( q \) instead of \( \dot{\omega} \) and \( \chi \). Thus,

\[ F + C = \frac{1}{2} \left[ 4(p^2 + q^2)(\dot{p}^2 + \dot{q}^2) + \dot{z}^2 \right] - \Omega + \frac{C}{2} = 0 \tag{7} \]

where

\[ \Omega = \frac{1}{2} (p^2 + q^2)^2 + (p^2 - q^2)\mu + \mu^2 + \frac{1 - \mu}{\sqrt{(p^2 + q^2)^2 + z^2}} + \frac{\mu}{\sqrt{(p^2 - q^2 + 1)^2 + 4pqq^2 + z^2}}. \tag{8} \]

Putting

\[ dt = 4r_1 d\tau \]

where

\[ r_1 = \sqrt{(p^2 + q^2)^2 + z^2}, \]

we get

\[ \begin{align*}
\frac{dp}{d\tau} &= 4r_1 \frac{\partial F}{\partial \dot{\omega}}, \quad \frac{dq}{d\tau} = 4r_1 \frac{\partial F}{\partial \dot{\chi}}, \quad \frac{dx_3}{d\tau} = 4r_1 \frac{\partial F}{\partial y_3} \\
\frac{d\omega}{d\tau} &= -4r_1 \frac{\partial F}{\partial p}, \quad \frac{d\chi}{d\tau} = -4r_1 \frac{\partial F}{\partial q}, \quad \frac{dy_3}{d\tau} = -4r_1 \frac{\partial F}{\partial x_3}
\end{align*} \tag{9} \]

Since

\[ r_1 \frac{\partial F}{\partial p} = \frac{\partial (r_1 F)}{\partial p} + \frac{2(p^2 + q^2)\rho C}{r_1} = \frac{\partial}{\partial p} [r_1 (F + C)] \]

and similarly for others, the Hamiltonian function reduces to

\[ H = 4r_1 (F + C) = 8(p^2 + q^2)(\dot{p}^2 + \dot{q}^2) r_1 + 2r_1 \dot{z}^2 - 4r_1 \Omega + 2r_1 C \]

i.e., to

\[ H = \frac{1}{2} \frac{p^2 + q^2}{r_1} \left\{ \left( \frac{dp}{d\tau} \right)^2 + \left( \frac{dq}{d\tau} \right)^2 \right\} + \frac{1}{8r_1} \left( \frac{dz}{d\tau} \right)^2 - 4r_1 \Omega + 2r_1 C \]
i.e., to

\[ H = \frac{1}{2} \frac{p^2 + q^2}{r_1} \left[ \left( \frac{dp}{d\tau} \right)^2 + \left( \frac{dq}{d\tau} \right)^2 + \frac{1}{4 \left( p^2 + q^2 \right)} \left( \frac{dz}{d\tau} \right)^2 \right. \]
\[ \left. - \frac{4r_1^2}{p^2 + q^2} (2\Omega - C) \right]. \] (10)

Now the transformation from \((x, y, z, \dot{x}, \dot{y}, \dot{z})\) to \((p, q, x_3, \alpha, \chi, y_3)\) will be said to regularize the solution if the Hamiltonian \(H\) in new variables (or in \(p, q, z, dp/d\tau, dq/d\tau, dz/d\tau\)) does not tend to infinity at the moment of collision, i.e., when \(r_1 \to 0\) [8, pages 54–67].

Let us examine the limit of \(H\) when \(r_1 \to 0\), i.e., when \(p \to 0\), \(q \to 0\) and \(z \to 0\). Putting \(\rho' = \sqrt{p^2 + q^2}\), \(\theta' = 2\tan^{-1} q/p\) in [4, 6.4], we shall have \(z = \mu (p^2 + q^2) h'_0(p, q, \mu)\) as seen in (16).

\[ \therefore \frac{p^2 + q^2}{r_1} = \frac{p^2 + q^2}{\sqrt{(p^2 + q^2)^2 + \mu^2 h'_0(p, q, \mu) (p^2 + q^2)^2}} \]
\[ = \frac{1}{\sqrt{1 + \mu^2 h'_0(p, q, \mu)}}. \]

Neglecting the terms of \(O(\mu^2)\), we find that

\[ \lim_{r_1 \to 0} \frac{p^2 + q^2}{r_1} = 1. \]

Assuming the value for \(dz/d\tau\) which will be shown in (21) in the form

\[ \frac{dz}{d\tau} = \mu \rho' h'(\rho', \theta', \mu) \]

we find that

\[ \lim_{r_1 \to 0} \frac{1}{4 \left( p^2 + q^2 \right)} \left( \frac{dz}{d\tau} \right)^2 \]
\[ = \lim_{r_1 \to 0} \frac{1}{4 \left( p^2 + q^2 \right)} \mu^2 (p^2 + q^2) h''_0(p, q, \mu) \]
\[ = 0(\mu^2) \text{ as } h'(p, q, \mu) \text{ is an analytic function in } \sqrt{p^2 + q^2} \text{ and } \mu. \]
Again,

\[\lim_{r_1 \to 0} \frac{4r_1^2}{p^2 + q^2} (2\Omega - C)\]

\[= \lim_{r_1 \to 0} \frac{4r_1}{p^2 + q^2} \cdot \lim_{r_1 \to 0} r_1 (2\Omega - C)\]

\[= \lim_{r_1 \to 0} \frac{8 \sqrt{(p^2 + q^2)^2 + z^2}}{p^2 + q^2} (1 - \mu) \text{ [from (8)]}\]

\[= 8 (1 - \mu) \lim_{r_1 \to 0} \frac{\sqrt{(p^2 + q^2)^2 + z^2}}{p^2 + q^2}\]

\[= 8 (1 - \mu). \quad (11)\]

\[\therefore \lim_{r_1 \to 0} H = \frac{1}{2} \left[ \left( \frac{dp}{d\tau} \right)^2 + \left( \frac{dq}{d\tau} \right)^2 - 8 (1 - \mu) \right]. \quad (12)\]

\[\therefore H \text{ does not tend to infinity at the moment of collision. It shows that the solution is not singular in the neighbourhood of the moment of collision.}\]

Let us consider the energy integral for the new Hamiltonian. Since H does not contain \( \tau \) explicitly, so

\[H' = \frac{\partial H}{\partial p} p' + \frac{\partial H}{\partial q} q' + \frac{\partial H}{\partial x_3} x_3' + \frac{\partial H}{\partial \omega} \omega' + \frac{\partial H}{\partial \lambda} \lambda' + \frac{\partial H}{\partial y_3^1} y_3^1\]

\[= \frac{\partial H}{\partial p} \cdot \frac{\partial H}{\partial \omega} + \ldots = 0.\]

\[\therefore H = \text{constant.}\]

As for \( r_1 \neq 0 \) any solution must satisfy the relation \( F + C = 0 \) and so H = 0 is the energy integral, i.e., for \( r_1 \neq 0 \), the energy integral or Jacobi's integral is

\[\left( \frac{dp}{d\tau} \right)^2 + \left( \frac{dq}{d\tau} \right)^2 + 4 \frac{1}{(p^2 + q^2)} \left( \frac{dz}{d\tau} \right)^2 - \frac{4r_1^2}{p^2 + q^2} (2\Omega - C) = 0 \quad (13)\]

and for \( r_1 = 0 \), the corresponding integral is

\[\left( \frac{dp}{d\tau} \right)^2 + \left( \frac{dq}{d\tau} \right)^2 = 8 (1 - \mu). \quad (13')\]

4. Phase Trajectories for \( \mu \neq 0 \) and Their Projections on the z-Plane

Let us find out the parametric equations of the periodic solution of our present problem in terms of \( p, q, z \). We have

\[p + iq = \sqrt{x - \mu} + iy\]
and

\[ x - \mu = \rho'^2 \cos \theta', \quad y = \rho'^2 \sin \theta' \]

whence

\[ p = \rho' \cos \frac{\theta'}{2}, \quad q = \rho' \sin \frac{\theta'}{2}, \quad z = z. \quad (14) \]

Thus combining with [4, 6.4] we get the required parametric representation of our solution. We have

\[ \rho^2 = \frac{1}{C} \left[ 1 + \frac{1}{C} \phi \left( \frac{1}{\sqrt{C}} \right) \right] \]

so for any trajectory the parameter is \( \theta \).

Let us find out a relation between \( p, q, z \) along an arbitrary trajectory. With this view let us solve the equations [4, 6.4] for \( \rho \) and \( \theta \), which can be evidently done for small \( \mu \). Then we shall get

\[
\begin{align*}
\rho &= \rho' \left[ \frac{1}{\sqrt{1 - \mu}} + \mu \rho'^2 f_6 (\rho', \theta', \mu) \right] \\
\theta &= \theta' + \mu \rho'^2 g_6 (\rho', \theta', \mu) \\
z &= \mu \rho'^2 h_6 (\rho', \theta', \mu)
\end{align*}
\]

where \( f_6, g_6, h_6 \) are analytic in \( \rho' \) and \( \mu \) and periodic in \( \theta' \) with the period \( 2\pi \). Also \( h_6 (\rho', \theta', \mu) \) is not identically zero as seen in Art. 2. Putting in the 1st and 3rd expressions of (15).

\[ \rho' = \sqrt{p^2 + q^2}, \quad \cos \theta' = \frac{p^2 - q^2}{p^2 + q^2}, \quad \sin \theta' = \frac{2pq}{p^2 + q^2}, \]

we get the equation to the family of trajectories under consideration in the form:

\[
\begin{align*}
p^2 + q^2 + \mu (p^2 + q^2)^2 \Phi (p, q, \mu) &= (1 - \mu) \rho^2 \\
z &= \mu (p^2 + q^2) h_6' (p, q, \mu)
\end{align*}
\]

where \( \Phi (p, q, \mu) \) and \( h_6' (p, q, \mu) \) are analytic functions in \( (p, q, \mu) \) and they are finite for \( p = q = z = 0 \). As

\[ C = \frac{1}{\rho^2 \left[ 1 + \rho^2 f_4 (\rho) \right]^2}, \]

to each \( \rho \) there will correspond a value of Jacobi's constant \( C \) for which (16)
represents the analytic continuation of the circular trajectory. Putting
\( \mu = 0 \) in (16), we get
\[
\begin{align*}
p^2 + q^2 &= \rho^2 \\
z &= 0
\end{align*}
\]   
(17)
which shows that \( \rho \) is the radius of the circular orbit for \( \mu = 0 \), i.e., \( \rho \) is the following double-valued function of \( C \):
\[
\rho = \begin{cases} 
-\sqrt{a_1(C)} \\
+\sqrt{a_2(C)}
\end{cases}
\]   
(18)
where \( a_1 \) and \( a_2 \) are the radii for the retrograde and the direct motion defined above. Taking \( \rho \) as a parameter for a fixed \( C \) in (16), we find that the equations (16) represent the retrograde and the direct orbit for \( \rho = -\sqrt{a_1} \) and \( \rho = \sqrt{a_2} \) which have been represented by Birkhoff by \( L_1 \) and \( L_2 \) in the plane restricted problem of three bodies [2].
\[
\begin{align*}
L_1: & \begin{cases} 
p^2 + q^2 + \mu(p^2 + q^2)^2 \Phi(p, q, \mu) = (1 - \mu) a_1 \\
z = \mu(p^2 + q^2) h_6'(p, q, \mu)
\end{cases} \\
L_2: & \begin{cases} 
p^2 + q^2 + \mu(p^2 + q^2)^2 \Phi(p, q, \mu) = (1 - \mu) a_2 \\
z = \mu(p^2 + q^2) h_6'(p, q, \mu)
\end{cases}
\]
For other \( \rho \) the curves (16) will not be trajectories for the value of \( C \) under consideration. But by virtue of the one-to-one correspondence of the transformation [4, 6.4] under the variation of \( \rho^2 \) in the region \( 0 < \rho^2 \leq r' \), where \( r' \) is the root of the equation
\[
v^2 = r^2 + \frac{2}{r} - C = 0 [4, \text{Art. 4 (Theorem)}]
\]
giving the zero velocity curve for \( \mu = 0 \) and \( C > 3 \) and it is such that \( \sigma^2 > 0 \) for \( 0 < r < r' \), the curves (16) are the mappings of the trajectories for the possible motion under the given \( C \) and \( \mu = 0 \). Corresponding to \( \rho^2 = 0 \) we shall get \( p = q = 0 = z \) for both \( \mu = 0 \) and \( \mu \neq 0 \). For \( \rho^2 = r' \) and \( \mu = 0 \), we shall get the curve of zero velocity. Let us find out that value of \( C \) for which the moving point for \( \mu = 0 \) always lies on the curve of zero velocity. For this the condition will be \( a_2(C) = r'(C) \) which is obvious from the graph of the function \( b(a) \) [4, Fig. 2]. But \( r' \) and \( a_2 \) satisfy the equations
\[
r'^2 + \frac{2}{r'} = C, \quad 2\sqrt{a_2} + \frac{1}{a_2} = C
\]
which in the region \( r' \geq 0, a_0 \geq 0, C \geq 3 \) have the unique solution \( r' = a = 1 \) for \( C = 3 \). Thus the required value of \( C \) is 3.

Since \( \frac{dC}{da} < 0 \) (taking into consideration that \( a_0 < 1 \)) and therefore the value of \( C \) for all circles \( p^2 + q^2 = r', z = 0 \) inside the circle of zero velocity \( p^2 + q^2 = r', z = 0 \), will always be greater than 3. Now we shall see that the values of \( C \) for the trajectories, representing the analytic continuation of the circular orbit for \( \mu \neq 0 \), are the same as for \( \mu = 0 \). Let us find out the value of \( C \) sufficiently near to the curve of zero velocity for \( \mu \neq 0 \). For it we shall find the value of \( C \) at a point of libration. We know that for the point of libration [7(ii)] whose co-ordinates are

\[
\begin{align*}
  x &= -1 + \sqrt{i}/3 + \ldots, \\
  y &= 0, \\
  z &= 0, \\
  C &= 3 (1 + \sqrt{3} \mu^2 + \ldots) + 0 (\mu)
\end{align*}
\]

whence it is clear that for sufficiently small \( \mu \) this value of \( C \) is greater than 3. Consequently, the orbits corresponding to the analytic continuation for \( \mu \neq 0 \) lie inside the circular cylinder \( r^2 = r', z = 0 \). Thus we find that the projections of the family of curves (16) on the \( z \)-plane cover the whole space of the possible motion for the variation of \( r^2 \) in the region

\[
0 \leq r^2 \leq r_1 (\mu, C)
\]

where

\[
a_0 (C) < r_1 (\mu, C) < r' (C)
\]

and

\[
r_1 (0, C) = r' (C).
\]

5. EQUATIONS OF THE MANIFOLD AND THE TRAJECTORIES ON IT

Let us consider the 6-dimensional phase-space \( \mathbb{R}_6 (p, p', q, q', z, z') \) where

\[
\begin{align*}
  \rho' &= \frac{dp}{d\tau}, \\
  q' &= \frac{dq}{d\tau}, \\
  z' &= \frac{dz}{d\tau}.
\end{align*}
\]

Jacobi's integral (13) defines a 5-dimensional sub-space in \( \mathbb{R}_6 \). Now let us consider the space \( G \) whose equation is obtained by differentiating

\[
p^2 + q^2 + \mu (p^2 + q^2)^2 \Phi (p, q, \mu) = (1 - \mu) r^2
\]

w.r. to \( \tau \), i.e.,

\[
G = pp' + qq' + \mu (p^2 + q^2) [A (p, q, \mu) p' + B (p, q, \mu) q'] = 0
\]
where
\[ A(p, q, \mu) = 2p \Phi(p, q, \mu) + \frac{1}{2} (p^2 + q^2) \frac{\partial \Phi}{\partial p} \]
and
\[ B(p, q, \mu) = 2q \Phi(p, q, \mu) + \frac{1}{2} (p^2 + q^2) \frac{\partial \Phi}{\partial q} . \]

Since \( C \) is involved only through \( \rho \) and as \( G \) is independent of \( \rho \) which shows that for any \( C \) the phase trajectory will lie on the surface \( G = 0 \). Thus we find that the phase trajectory lies on a four-dimensional surface and through any point on this surface there will pass a trajectory. The equations to the trajectory which lies on this surface can be written in the form:

\[
\begin{align*}
p &= \rho' \cos \frac{\theta'}{2} \\
q &= \rho' \sin \frac{\theta'}{2} \\
z &= \mu \rho' \theta (p', q', \mu) \\
p' &= -\sqrt{\frac{4 (2\Omega - C) r_1^2}{p^2 + q^2}} - \frac{1}{4 (p^2 + q^2)} \left( \frac{dz}{d\tau} \right)^2 \sin \frac{\theta''}{2} \\
q' &= \sqrt{\frac{4 (2\Omega - C) r_1^2}{p^2 + q^2}} - \frac{1}{4 (p^2 + q^2)} \left( \frac{dz}{d\tau} \right)^2 \cos \frac{\theta''}{2} \\
z' &= \mu \rho' \theta (p', q', \mu)
\end{align*}
\]

The first five expressions are obvious from (13), (14) and (15). Let us see the expression for \( z' \). We have

\[
z = \mu (p^2 + q^2) h_\theta (p, q, \mu)
\]

where \( h_\theta (p, q, \mu) \) is an analytic function of \( p, q \) and \( \mu \). Now

\[
\frac{dz}{d\tau} = z_pp' + z_qq'
\]

and squaring it and substituting the values of \( p' \) and \( q' \) from (21) we shall get

\[
\left( \frac{dz}{d\tau} \right)^2 = \frac{4 (2\Omega - C) r_1^2}{p^2 + q^2} \times \frac{\left( z_p \sin \frac{\theta''}{2} - z_q \cos \frac{\theta''}{2} \right)^2}{1 + \left( z_p \sin \frac{\theta''}{2} - z_q \cos \frac{\theta''}{2} \right)^2/4 (p^2 + q^2)}.
\]
From Art. 4 and (22) it is clear that if \( p \to 0 \) and \( q \to 0 \), then \( z \to 0 \) and so \( r_1 \to 0 \) for \( r_1 = \sqrt{(p^2 + q^2) + z^2} \).

Now the only singularity of \( dz/d\tau \) appears through the expression

\[
\frac{4(2\Omega - C) r_1^2}{p^2 + q^2} \frac{(z_p \sin \frac{\theta''}{2} - z_q \cos \frac{\theta''}{2})^2}{1 + (z_p \sin \frac{\theta''}{2} - z_q \cos \frac{\theta''}{2})^2 / 4 (p^2 + q^2)}
\]

to be the point \( p = q = z = 0 \). We shall now show that

\[
\lim_{(p, q, z) \to (0, 0, 0)} \left( \frac{dz}{d\tau} \right)^2
\]

exists and it is finite. Since

\[
\lim_{(p, q, z) \to (0, 0, 0)} \left( \frac{dz}{d\tau} \right)^2 = \lim_{(p, q, z) \to (0, 0, 0)} \frac{4(2\Omega - C) r_1^2}{p^2 + q^2} \times \lim_{(p, q, z) \to (0, 0, 0)} \frac{(z_p \sin \frac{\theta''}{2} - z_q \cos \frac{\theta''}{2})^2}{1 + (z_p \sin \frac{\theta''}{2} - z_q \cos \frac{\theta''}{2})^2 / 4 (p^2 + q^2)}
\]

From Art. 3, it is known that

\[
\lim_{(p, q, z) \to (0, 0, 0)} \frac{4(2\Omega - C) r_1^2}{p^2 + q^2} = 8 (1 - \mu).
\]

To evaluate

\[
\lim_{(p, q, z) \to (0, 0, 0)} \frac{(z_p \sin \frac{\theta''}{2} - z_q \cos \frac{\theta''}{2})^2}{1 + (z_p \sin \frac{\theta''}{2} - z_q \cos \frac{\theta''}{2})^2 / 4 (p^2 + q^2)}
\]

let us make a radial approach to the origin, and for it put \( p = \rho' \cos \theta'/2 \), \( q = \rho' \sin \theta'/2 \), then

\[
1 + \frac{(z_p \sin \frac{\theta''}{2} - z_q \cos \frac{\theta''}{2})^2}{4 (p^2 + q^2)}
\]
Thus

\[
\lim_{(\rho, \theta, \varphi) \to (0, 0, 0)} \left( z_p \sin \frac{\theta''}{2} - z_q \cos \frac{\theta''}{2} \right)^2 = 0.
\]

Similarly,

\[
\lim_{(\rho, \theta, \varphi) \to (0, 0, 0)} \left( \frac{dz}{d\tau} \right)^2 = 8 (1 - \mu) \times 0 = 0.
\]

Thus the limit exists and it shows that \( dz/d\tau \) is analytic function of \( \rho, q \) and \( \mu \). As \( dz/d\tau = 0 \) for \( \rho' = \mu = 0 \), so

\[
\frac{dz}{d\tau} = \mu \rho' \tilde{f}(\rho', \theta', \mu)
\]

where \( \tilde{f}(\rho', \theta', \mu) \) is an analytic function of \( \rho' \) and \( \mu \) and periodic in \( \theta' \) with the period \( 2\pi \).

In this way all the co-ordinates of a point on the manifold are expressed in terms of the parameters \( \rho', \theta', \) and \( \theta'' \). For a fixed \( \rho \), we shall show now that all the parameters \( \rho', \theta' \) and \( \theta'' \) involved in (21) can be expressed in terms of a single parameter \( \theta \).

Through [4, 6.4] we get

\[
\rho' = \rho \left[ \sqrt{1 - \mu} + \mu \rho^2 f(\rho, \theta, \mu) \right],
\]

\[
\theta' = \theta + \mu \rho^2 g(\rho, \theta, \mu)
\]

By (21) we find

\[
\theta'' = \arctan \left( -\frac{\rho'}{q} \right)
\]

\[
= \theta' + 2 \arctan \left[ \frac{\mu \rho' \left( \beta \cos \frac{\theta'}{2} - A \sin \frac{\theta'}{2} \right)}{1 + \mu \rho' \left( A \cos \frac{\theta'}{2} + B \sin \frac{\theta'}{2} \right)} \right]
\]

taking the help of the equation \( G = 0 \).
Each curve of the family is obtained by the variation of \( \theta \) in the range \( 0 \leq \theta < 2\pi \). Thus we find that for each value of \( \rho^2 \) from the range
\[
0 \leq \rho^2 \leq r_1(\mu, C)
\]
\( i.e., \) for each value of \( \rho \) from the range
\[
-\sqrt{r_1(\mu, C)} \leq \rho \leq \sqrt{r_1(\mu, C)}
\]
the equations (21) define a unique curve on the 5-dimensional surface (13).

The trajectories are seen to satisfy the following set of equations also:
\[
\begin{align*}
F & = p'^2 + q'^2 + \frac{1}{4(p^2 + q^2)} z'^2 - \frac{4r_1^2}{p^2 + q^2} (2\Omega - C) = 0 \\
G & = pp' + qq' + \mu (p^2 + q^2) \left[ A(p, q, \mu) p' + \beta(p, q, \mu) q' \right] = 0 \\
p^2 + q^2 + \mu (p^2 + q^2)^2 \Phi(p, q, \mu) & = (1 - \mu) \rho^2 \\
z & = \mu \rho^2 h_b(p', q', \mu) = (p^2 + q^2) h_b'(p, q, \mu) \\
z' & = \mu \rho h_b(p', q', \mu) = \mu \sqrt{p^2 + q^2} h_b'(p, q, \mu)
\end{align*}
\] (28)


From the previous article it is clear that the trajectories for the variation of \( \rho \) between \( -\sqrt{r_1(\mu, C)} \) and \( + \sqrt{r_1(\mu, C)} \), where \( r_1(\mu, C) \) is defined by (19), will generate a surface on the 5-dimensional manifold. As \( z \) and \( z' \) can always be expressed in terms of \( p, q \) and \( \mu \), \( R_5 \) reduces to \( R_3(p, q, p') \) or \( R_3(p, q, q') \).

Definition.—Imagine a circle in the \( zx \)-plane with its centre at an arbitrary point \( (x, 0, 0) \) on the \( x \)-axis. The surface of revolution made by this circle about the line \( z'Oz \) is called the 3-dimensional torus.

Equation of the torus.—Let \( DE \) be the section of the torus by a plane through a point \( P(x, y, z) \) and the axis \( Oz \) and let \( AA' \) and \( BB' \) be boundaries of the ring of section made by the \( z \)-plane with the torus. Suppose that \( A \) is the inner radius and \( A + r' \), the outer radius of the ring where \( r' \) is given as in Art. 4, whence it is clear that \( r' \) is a constant depending on \( C \). It is related with \( r_1(\mu, C) \) by (20). Let \( A + \rho^2, \theta, \tau \) be the cylindrical co-ordinates of the point \( P(x, y, z) \), then \( x = (A + \rho^2) \cos \theta, y = (A + \rho^2) \sin \theta \). Making use of the property that every point \( P(x, y, z) \) lies on a circle \( DE \) of diameter \( r' \) and \( z \) is the perpendicular from \( P \) on the diameter, we find that
\[
z = \rho \sqrt{r' - \rho^2}.
\]
Thus the parametric representation of the torus is
\[ x = (A + \rho^2) \cos \theta, \quad y = (A + \rho^2) \sin \theta, \quad z = \rho \sqrt{r' - \rho^2} \]  

(29)

where \( \rho, \theta \) vary over the region

\[
\begin{align*}
-\sqrt{r'} \leq \rho & \leq \sqrt{r'} \\
0 \leq \theta & \leq 2\pi
\end{align*}
\]

(30)

and \( \rho (> / < 0) \) according as \( z (> / < 0) \) and thus for each fixed \( \rho \) over (30), there will correspond a circle \( T \) as shown in the figure below.

It is clear that the region

\[
\begin{align*}
-\sqrt{r_1(\mu, C)} \leq \rho & \leq \sqrt{r_1(\mu, C)} \\
0 \leq \theta & < 2\pi
\end{align*}
\]

(31)

in which our trajectory (21) is defined, is a part of (30). Comparing the points of the torus and our manifold for the same \((\rho, \theta)\), we find that each curve given by (21) corresponds to a circle \( T \) for a fixed \( \rho \). In particular, to each \( \rho^2 \) from the region \( 0 < \rho^2 < r_1(\mu, C) \) there will correspond two curves giving direct \((\rho > 0)\) and retrograde \((\rho < 0)\) motion and on the torus two circles

\[
\begin{align*}
x = (A + \rho^2) \cos \theta & \quad \text{and} \quad x = (A + \rho^2) \cos \theta \\
y = (A + \rho^2) \sin \theta & \quad \text{and} \quad y = (A + \rho^2) \sin \theta \\
z = \rho \sqrt{r' - \rho^2} & \quad \text{and} \quad z = -\rho \sqrt{r' - \rho^2}
\end{align*}
\]

(32)
Since for $p = 0, p' = 0$ and $\theta' = \theta'' = 0$ which is obvious from (24) and (25), so for $\rho = 0$ from (21)

\[
p = 0, \; q = 0, \; z = 0, \; p' = -\sqrt{K} \sin \frac{\theta}{2}, \; q' = \sqrt{K} \cos \frac{\theta}{2}, \; z' = 0
\]  

(33)

where

\[
K = \left[ \frac{4(2\Omega - C) r_1^2}{p^2 + q^2} - \frac{1}{4(p^2 + q^2)} \left( \frac{dz}{d\tau} \right)^2 \right]_{p=0} \\
= \left[ \frac{4(2\Omega - C) r_1^2}{p^2 + q^2} \right]_{p=0} - \left[ \frac{1}{4(p^2 + q^2)} \left( \frac{dz}{d\tau} \right)^2 \right]_{p=0}.
\]

As $\rho = 0$ implies $p = q = z = 0$ and vice-versa, so in evaluation of $K$ we shall take $p = 0, q = 0, z = 0$ instead of $\rho = 0$. As for $p = q = z = 0$, $K$ is undefined, hence by the value of $K$ at $p = q = z = 0$, we shall mean its limiting value. Now as

\[
\lim_{(p, q, z) \to (0, 0, 0)} K = \lim \left[ \frac{4(2\Omega - C) r_1^2}{p^2 + q^2} \right] - \lim \frac{4(2\Omega - C) r_1^2}{p^2 + q^2} \times \frac{\left( \frac{\theta''}{2} - \frac{z \cos \frac{\theta''}{2}}{4(\rho^2 + q^2)} \right)^2}{1 + \lim \frac{\left( \frac{\theta''}{2} - \frac{z \cos \frac{\theta''}{2}}{4(\rho^2 + q^2)} \right)^2}{4(\rho^2 + q^2)}}
\]

[using the expression for $(dz/d\tau)^2$ from Art. 5].

From Art. 5, we also know that

\[
\lim \frac{4(2\Omega - C)}{p^2 + q^2} = 8(1 - \mu)
\]

and

\[
\lim \frac{\left( \frac{\theta''}{2} - \frac{z \cos \frac{\theta''}{2}}{4(\rho^2 + q^2)} \right)^2}{4(\rho^2 + q^2)} = \lim_{\theta'' \to \theta'} \frac{1}{4\mu^2 h'_{e^2} \sin^2 \frac{\theta''}{2}} = 0.
\]

\[
\therefore \; \lim K = 8(1 - \mu) - 8(1 - \mu) \times \frac{0}{1 + 0} = 8(1 - \mu).
\]

Putting this limiting value of $K$ in (33), we find that to the value $\rho = 0$ there corresponds the unique trajectory for both $\mu = 0$ and as well as for $\mu \neq 0$, whose equations are
\[ p = q = z = 0, \quad p' = -2 \sqrt{2(1-\mu)} \sin \frac{\theta}{2}, \]

\[ q' = 2 \sqrt{2(1-\mu)} \cos \frac{\theta}{2}, \quad z' = 0 \]  

(34)

and the unique circle

\[ x = A \cos \theta, \quad y = A \sin \theta, \quad z = 0 \]

on the torus. We have already seen in Art. 4 that the motion is not defined for \( \rho^2 = r' \) and \( \mu = 0 \), but, however, on the torus (29) there will correspond a unique circle \( C \):

\[ x = (A + r') \cos \theta, \quad y = (A + r') \sin \theta, \quad z = 0. \]

For \( \mu = 0 \), corresponding to \( \rho^2 = r' = r_1(0, C) \) from [4, 4.3]

\[ \begin{align*}
4 \frac{(2\Omega - C) r_1^2}{p^2 + q^2} & - \frac{1}{4(p^2 + q^2)} \left( \frac{dz'}{dt} \right)^2 = 4 \frac{(2\Omega - C)(p^2 + q^2)}{p^2 + q^2} \\
& = 4(r'^2 - Cr' + 2) = 0
\end{align*} \]

and so the equations to the trajectories \( L_1 \) and \( L_2 \) from (21) are respectively

\[ \begin{align*}
p &= \sqrt{r'} \cos \frac{\theta}{2}, \quad q = \sqrt{r'} \sin \frac{\theta}{2}, \quad z = 0 = p' = q' = z' \\
n &= -\sqrt{r'} \cos \frac{\theta}{2}, \quad q = \sqrt{r'} \sin \frac{\theta}{2}, \quad z = 0 = p' = q' = z'
\end{align*} \]

(35)

(36)

which have been represented uniquely for one is obtained from the other by putting \( \theta + 2\pi \) for \( \theta \) and this latter trajectory gives the curve of zero velocity.

Thus we find that there is a one-to-one correspondence between the surface (21) and the torus (29) for \( \mu = 0 \) and for \( \mu > 0 \) this one-one correspondence is preserved between the surface (21) and a part of the surface of the torus (29) defined by the region (31).

7. Birkhoff's Rings for \( \mu \neq 0 \)

The surface of the torus (29) can be divided into the rings defined by two regions in which (30) can be divided, i.e.,

\[ \begin{align*}
-\sqrt{a_1(C)} & \leq \rho \leq \sqrt{a_9(C)} \\
0 & \leq \theta < 2\pi
\end{align*} \]

(37)
and
\[ -\sqrt{r'(C)} \leq \rho < -\sqrt{a_1(C)}, \quad \sqrt{a_2(C)} < \rho \leq \sqrt{r'(C)} \quad \{ \]
\[ 0 \leq \theta < 2\pi \]
\] (38)

The general boundaries of these regions are the lines \( L_1 \) and \( L_2 \) defined by \( \rho = -\sqrt{a_1(C)} \) and \( \rho = \sqrt{a_2(C)} \) which belong to (37). Here \( x = A \cos \theta, \ y = A \sin \theta, \ z = 0 \) is a circle belonging to (37) and \( x = (A + r') \cos \theta, \ y = (A + r') \sin \theta, \ z = 0 \) belongs to (38).

Let us consider the circular ring on the \( xy \)-plane in the form
\[ x = (A + \rho) \cos \theta, \ y = (A + \rho) \sin \theta \quad (39) \]
where \( \rho \) and \( \theta \) vary in the region (37). It is easy to establish one-to-one correspondence between the ring (39) and the ring on the torus in the range (37). Then \( L_1 \) and \( L_2 \) correspond to
\[
\begin{align*}
L_1 : & \quad x = (A - \sqrt{a_1}) \cos \theta \\
& \quad y = (A - \sqrt{a_1}) \sin \theta \\
L_2 : & \quad x = (A + \sqrt{a_2}) \cos \theta \\
& \quad y = (A + \sqrt{a_2}) \sin \theta 
\end{align*}
\]

The ring with the boundaries \( L_1 \) and \( L_2 \) is called Birkhoff’s ring and thus it is shown that there is a one-to-one correspondence between the trajectories for \( \mu \neq 0 \) and the circles included between \( L_1 \) and \( L_2 \).

The phase trajectory for the collision corresponds to
\[ x = A \cos \theta, \ y = A \sin \theta. \]
Thus corresponding to each trajectory given by (29) we shall have a circle on the \( xy \)-plane. It is to be marked here that only due to Levi-Civita’s transformation the phase trajectories could be represented without singularity.

8. ACKNOWLEDGEMENT

I shall take this opportunity to offer my hearty thanks to Dr. N. S. Nagendra Nath for his valuable time to go through this paper.

9. REFERENCES

3. Birkhoff, G. D.  
   "Surface transformations and their dynamical application,"  

4. Choudhry, R. K.  

5. Levi-Civita  

6. Merman, G. A.  

7. Moulton  
   (i) *Periodic Orbits* (ii) *An Introduction to Celestial Mechanics*.

8. Siegel  
   *Vorlesungen uber Himmelsmechanic*, Berlin, Gottingen,  
   (Russian Translation, pp. 54–67, 90–102).