FLOW OF A VISCOUS INCOMPRESSIBLE FLUID BETWEEN TWO PARALLEL PLATES, ONE IN UNIFORM MOTION AND THE OTHER AT REST, WITH SUCTION AT THE STATIONARY PLATE

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INTRODUCTION

THE exact solution of Navier-Stokes equations for the plane Coutte flow between two plates, one at rest and the other in uniform motion, is already known.[1] The velocity distribution between the plates is linear with the maximum velocity at the plate in motion.

In this paper an attempt has been made to find the solution of the Navier-Stokes equations for the flow of a viscous incompressible fluid between two plates, one at rest and the other in uniform motion, with small uniform suction at the sationary plate. A solution has been obtained under the assumption that the pressure between the two plates is uniform. It has been shown that due to suction a linear transverse velocity is superimposed over the longitudinal velocity. With suction the longitudinal velocity distribution between the plates becomes parabolic and decreases along the length of the plate. The longitudinal velocity, the shearing stress at the stationary plate, and the volume rate of flow increase with σ $(= v_0 y_0/\nu)$, the suction parameter defined with reference to the suction velocity and the distance between the two plates.

NOTATIONS

x = co-ordinate along the length of the plate.

y = co-ordinate normal to the plate measured from the plate at rest,

z = co-ordinate perpendicular to the plane of x and y,

u = velocity in x direction.

v = velocity in y direction.

w = velocity in z direction.

U = uniform velocity of the plate in motion.

ρ = density of the fluid.

p = pressure.

v_0 = uniform suction velocity at the stationary plate.

y_0 = distance between the two plates.

μ = coefficient of viscosity.

ν $= \frac{\mu}{\rho}$ kinematic viscosity.

η $= \frac{y}{y_0}$ dimensionless y co-ordinate.

σ $= \frac{v_0 y_0}{\nu}$ suction parameter.

ξ $= (1 - \eta)\sqrt{\sigma}$.

$$S_0 = \sum_0^{\infty} \frac{1}{2^n (2n-1) \angle n} \sigma^n.$$

$$S = \sum_0^{\infty} \frac{1}{2^n (2n-1) \angle n} \sigma^n (1-\eta)^{2n-1}.$$

τ_0 $= \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$, shearing stress at the stationary plate.

Q $= \int_0^{y_0} u\,dy$, volume rate of longitudinal flow.

Q_0 = volume rate of longitudinal flow for solid plates ($\sigma = 0$).

C_Q $= \frac{Q}{Q_0}$, flow coefficient.

Q′ = volume of the fluid removed through suction.

Equations of Motion and Their Solution

The Navier-Stokes equations of motion of viscous incompressible fluids in cartesian rectangular co-ordinates are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{2}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{3}$$

and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{4}$$

For the flow of a viscous incompressible fluid between two parallel plates, one in uniform motion and the other at rest, it is assumed that

$\frac{\partial p}{\partial x}=0$, since the flow is due to shear only,

and

$\frac{\partial p}{\partial y}=0$, since the pressure variation at right ang'es to the plate is neglected for small suction.

Further

$$\left.\begin{aligned} w &= 0 \\ \frac{\partial}{\partial z} &= 0 \end{aligned}\right\} \text{ for two-dimensional flow.}$$

Hence Equation (3) identically vanishes and Equations (1), (2) and (4) become

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{5}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{6}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{7}$$

If the axis of x is taken along the plate at rest and y is measured at right angles to it, the boundary conditions are:

$$\begin{aligned} &y=0, \quad v=-v_0, \quad u=0 \\ &y=y_0, \quad v=0, \quad u=U \end{aligned} \tag{8}$$

where y_0 is the distance between the two plates, v_0 is the uniform suct'on at the plate at rest and U is the velocity of the plate in motion.

Because of uniform suction $\partial v/\partial x = 0$ and the equation of motion in y direction becomes

$$v \frac{\partial v}{\partial y} = \nu \frac{\partial^2 v}{\partial y^2} \tag{9}$$

with the boundary conditions

$$\begin{aligned} v &= -v_0 \quad \text{when} \quad y = 0 \\ v &= 0 \quad \text{when} \quad y = y_0. \end{aligned} \tag{10}$$

I` the suction volocity is assumed to be very small, the term on the left-hand side of equation (9) is very small and the right-hand side must therefore be similarly small and approximately equal to zero. Then,

$$\frac{\partial^2 v}{\partial y^2} = 0. \tag{11}$$

The solution of Equation (11) with conditions (10) is

$$v = v_0\left(\frac{y}{y_0} - 1\right) \tag{12}$$

or

$$\frac{v}{-v_0} = 1 - \eta,$$

where

$$\eta = \frac{y}{y_0}.$$

The distribution of transverse velocity has been shown in Fig. 1.

From Equation (7),

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = \frac{-v_0}{y_0}. \tag{13}$$

Hence

$$\frac{\partial^2 u}{\partial x^2} = 0. \tag{14}$$

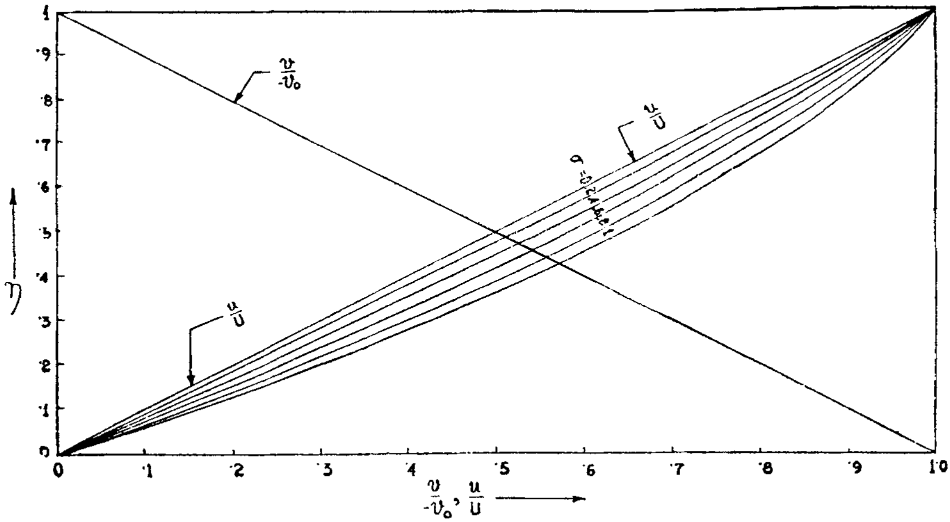

FIG. 1. Shear flow between two walls, one in uniform motion and the other at rest, with uniform suction at the stationary wall. Longitudinal and transverse velocity profiles plotted against η.

Substituting from equations (12), (13) and (14) into Equation (5) we get

$$\nu \frac{\partial^2 u}{\partial y^2} - v_0\left(\frac{y}{y_0} - 1\right)\frac{\partial u}{\partial y} + \frac{v_0}{y_0} u = 0$$

or

$$\frac{\partial^2 u}{\partial \eta^2} - \sigma(\eta - 1)\frac{\partial u}{\partial \eta} + \sigma u = 0 \tag{15}$$

where

$$\sigma = \frac{v_0 y_0}{\nu}$$

with the conditions

$$u = 0 \quad \text{when} \quad \eta = 0$$

and

$$u = \mathrm{U} \quad \text{when} \quad \eta = 1. \qquad 16)$$

Let

$$\xi = (1 - \eta)\sqrt{\sigma}.$$

Then Equation (15) changes to

$$\frac{\partial^2 u}{\partial \xi^2} - \xi \frac{\partial u}{\partial \xi} + u = 0 \tag{17}$$

and the conditions (16) become

$$u = 0 \quad \text{when} \quad \xi = \sqrt{\sigma}$$

and

$$u = \text{U} \quad \text{when} \quad \xi = 0. \tag{18}$$

The series solution of Equation (17) is

$$u = a_1 \xi - a_0 \sum_{0}^{\infty} \frac{1}{2^n (2n-1) \angle n} \xi^{2n}. \tag{19}$$

Applying the conditions (18), we have

$$a_0 = \text{U}$$

and

$$a_1 = \frac{\text{U}}{\sqrt{\sigma}} \sum_{0}^{\infty} \frac{1}{2^n (2n-1) \angle n} \sigma^n.$$

Hence,

$$\frac{u}{\text{U}} = (1 - \eta)(\text{S}_0 - \text{S}) \tag{20}$$

where

$$\text{S}_0 = \sum_{0}^{\infty} \frac{1}{2^n (2n-1) \angle n} \sigma^n,$$

$$\text{S} = \sum_{0}^{\infty} \frac{1}{2^n (2n-1) \angle n} \sigma^n (1 - \eta)^{2n-1}.$$

For

$$\sigma = 0, \quad \frac{u}{\text{U}} = \eta = \frac{y}{y_0}$$

as in the case of plane Coutte motion without suction. The longitudinal ve'ocity has been calculated for various values of η and σ and the results of calculation are shown in Table I and have been plotted in Fig. 1.

TABLE I

Values of $u/U = (1 - \eta)(S_0 - S)$ for various values of η and σ

$\eta \rightarrow$ / $\sigma \downarrow$	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·000
0·0	0·000	0·100	0·200	0·300	0·400	0·500	0·600	0·700	0·800	0·900	1·000
0·2	0·000	0·109	0·217	0·322	0·425	0·526	0·625	0·722	0·816	0·909	1·000
0·4	0·000	0·120	0·235	0·345	0·444	0·553	0·651	0·744	0·833	0·919	1·000
0·6	0·000	0·133	0·254	0·371	0·482	0·582	0·677	0·768	0·851	0·929	1·000
0·8	0·000	0·143	0·276	0·398	0·509	0·613	0·707	0·793	0·870	0·939	1·000
1·0	0·000	0·157	0·299	0·427	0·542	0·645	0·737	0·819	0·890	0·950	1·000

The shearing stress at the wall is

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= -\frac{\mu U}{y_0}\left[\frac{\partial}{\partial(1-\eta)}\{(1-\eta)(S_0 - S)\}\right]_{1-\eta=1}$$

i.e.,

$$\frac{\tau_0 y_0}{\mu U} = \sum_0^\infty \frac{1}{2^n \angle n}\sigma^n. \tag{21}$$

The discharge per unit breadth of the plates is

$$Q = \int_0^{y_0} u\,dy = U y_0 \int_0^1 (S_0 - S)\,Z\,dZ$$

where $Z = 1 - \eta$ and u/U is a function of Z.

Hence

$$Q = U y_0 \int_0^1 \left[S_0 Z - \sum_0^\infty \frac{1}{2^n (2n-1)\angle n}\sigma^n Z^{2n}\right] dZ$$

$$= U y_0 \left[\frac{S_0}{2} - \sum_0^\infty \frac{1}{2n(2n-1)(2n+1)\angle n}\sigma^n\right]. \tag{22}$$

The discharge for solid plates ($\sigma = 0$) is

$$Q_0 = \frac{Uy_0}{2}. \qquad (23)$$

The discharge coefficient is

$$C_Q = \frac{Q}{Q_0} = \left[S_0 - \sum_0^{\infty} \frac{1}{2^{n-1}(2n-1)(2n+1)\angle n} \sigma^n \right]$$

$$= \sum_0^{\infty} \frac{1}{2^n (2n+1)\angle n} \sigma^n. \qquad (24)$$

The shearing stress at the stationary plate and the flow coefficient have been calculated for various values of σ and the results of calculations have been shown in Table II and plotted in Fig. 2.

TABLE II

S_0, $\tau_0 y_0/\mu U$ and C_Q for various values of $\sigma = \frac{v_0 y_0}{\nu}$

σ	S_0	$\tau_0 y_0/\mu U$	C_Q
0·000	−1·000	1·000	1·000
0·100	−0·950	1·051	1·017
0·200	−0·898	1·105	1·034
0·300	−0·846	1·162	1·052
0·400	−0·793	1·221	1·071
0·500	−0·739	1·284	1·090
0·600	−0·684	1·350	1·110
0·700	−0·628	1·419	1·130
0·800	−0·571	1·492	1·151
0·900	−0·513	1·568	1·173
1·000	−0·454	1·649	1·195

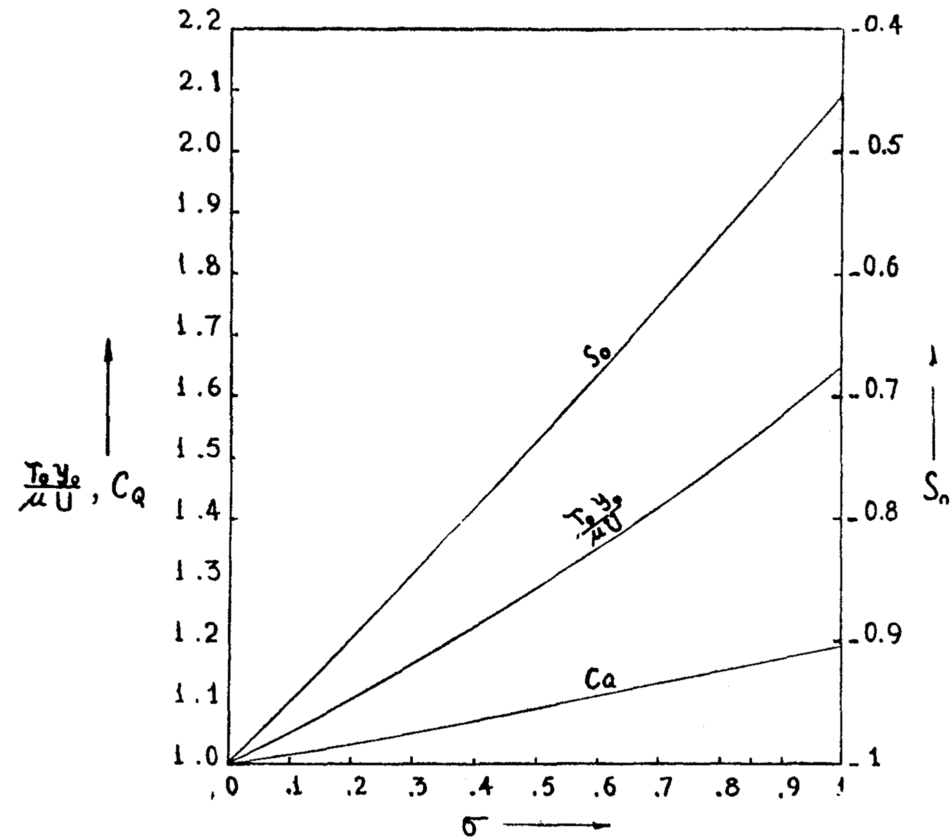

FIG. 2. Shear flow between two walls, one in uniform motion and the other at rest, with uniform suction at the stationary wall.

S_0, $\tau_0 y_0/\mu U$ and C_Q plotted against $\sigma = v_0 y_0/\nu$

The volume of fluid removed through suction is given by

$$Q' = -\, v_0 \,.\, A = \frac{-\, v_0}{U} \,.\, AU$$

where A is the wetted area of the stationary plate.

CONCLUSIONS

The following results have been obtained for the shear flow between two plates, with uniform suction at the stationary plate.

(*a*) The longitudinal velocity distribution

$$\frac{u}{U} = (1 - \eta)\,(S_0 - S),$$

(*b*) Transverse velocity distribution

$$\frac{v}{-v_0} = 1 - \eta.$$

(*c*) Shearing stress at the wall

$$\frac{\tau_0 y_0}{\mu U} = \sum_0^\infty \frac{1}{2^n \angle n} \sigma^n.$$

(*d*) Flow coefficient

$$C_Q = \sum_0^\infty \frac{1}{2^n (2n+1) \angle n} \sigma^n.$$

As a consequence of small suction at the stationary wall, a linear transverse velocity is superimposed over the longitudinal velocity due to shear. The longitudinal velocity distribution between the two plates becomes parabolic for small velocities of suction. The velocity u and the flow coefficient C_Q increase as σ increases. The longitudinal velocity and the volume rate of flow decrease along the length of the wall.

Proceeding to the limit $\sigma = 0$, we have

$$\frac{u}{U} = \frac{y}{y_0}$$

$$v = 0$$

$$\frac{\tau_0 y_0}{\mu U} = 1$$

and

$$C_Q = 1$$

as is the plane Coutte flow between two plates. Since the volume flow is decreasing all along the wall, the Reynolds number is similarly decreasing and if the flow is laminar at the leading edge, it will remain so. The ever-decreasing longitudinal velocity and the uniform suction velocity will at some point lead to v_0 being of comparable magnitude to u and the solution may cease to hold good under these conditions.

Summary

In this paper an attempt has been made to find the solution of the Navier-Stokes equations for the flow of a viscous incompressible fluid between two

plates, one at rest and the other in uniform motion, with small uniform suction at the stationary plate. A solution has been obtained under the assumption that the pressure between the two plates is uniform. It has been shown that due to suction a linear transverse velocity is superimposed over the longitudinal velocity. With suction, the longitudinal velocity distribution between the plates becomes parabolic and decreases along the length of the plate.

The longitudinal velocity, the shearing stress at the stationary plate, and the volume rate of flow increase with σ $(= v_0 y_0/\nu)$, the suction parameter defined with reference to the suction velocity, and the distance between the two plates. For $\sigma = 0$ the results transform to the known results for plane Coutte flow without suction.

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