FLOW OF A VISCOUS INCOMPRESSIBLE FLUID BETWEEN TWO PARALLEL PLATES, ONE IN UNIFORM MOTION AND THE OTHER AT REST, WITH SUCTION AT THE STATIONARY PLATE

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INTRODUCTION

The exact solution of Navier-Stokes equations for the plane Couette flow between two plates, one at rest and the other in uniform motion, is already known. The velocity distribution between the plates is linear with the maximum velocity at the plate in motion.

In this paper an attempt has been made to find the solution of the Navier-Stokes equations for the flow of a viscous incompressible fluid between two plates, one at rest and the other in uniform motion, with small uniform suction at the stationary plate. A solution has been obtained under the assumption that the pressure between the two plates is uniform. It has been shown that due to suction a linear transverse velocity is superimposed over the longitudinal velocity. With suction the longitudinal velocity distribution between the plates becomes parabolic and decreases along the length of the plate. The longitudinal velocity, the shearing stress at the stationary plate, and the volume rate of flow increase with \( \sigma (= v_0 / v) \), the suction parameter defined with reference to the suction velocity and the distance between the two plates.

NOTATIONS

\[ x = \text{co-ordinate along the length of the plate.} \]
\[ y = \text{co-ordinate normal to the plate measured from the plate at rest,} \]
\[ z = \text{co-ordinate perpendicular to the plane of } x \text{ and } y. \]
Flow of a Viscous Incompressible Fluid between Two Parallel Plates

\[ u = \text{velocity in } x \text{ direction.} \]
\[ v = \text{velocity in } y \text{ direction.} \]
\[ w = \text{velocity in } z \text{ direction.} \]
\[ U = \text{uniform velocity of the plate in motion.} \]
\[ \rho = \text{density of the fluid.} \]
\[ p = \text{pressure.} \]
\[ v_0 = \text{uniform suction velocity at the stationary plate.} \]
\[ y_0 = \text{distance between the two plates.} \]
\[ \mu = \text{coefficient of viscosity.} \]
\[ \nu = \frac{\mu}{\rho} \text{ kinematic viscosity.} \]
\[ \eta = \frac{y}{y_0} \text{ dimensionless } y \text{ co-ordinate.} \]
\[ \sigma = \frac{v_0 y_0}{\nu} \text{ suction parameter.} \]
\[ \xi = (1 - \eta) \sqrt{\sigma}. \]
\[ S_0 = \sum_{n=0}^{\infty} \frac{1}{2^n (2n - 1)} \zeta_n \sigma^n. \]
\[ S = \sum_{n=0}^{\infty} \frac{1}{2^n (2n - 1)} \zeta_n \sigma^n (1 - \eta)^{2n-1}. \]
\[ \tau_0 = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}, \text{ shearing stress at the stationary plate.} \]
\[ Q = \int_0^y u dy, \text{ volume rate of longitudinal flow.} \]
\[ Q_0 = \text{volume rate of longitudinal flow for solid plates (} \sigma = 0). \]
\[ C_0 = \frac{Q}{Q_0}, \text{ flow coefficient.} \]
\[ Q' = \text{volume of the fluid removed through suction.} \]

**Equations of Motion and Their Solution**

The Navier-Stokes equations of motion of viscous incompressible fluids in cartesian rectangular co-ordinates are
and the equation of continuity is
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \] (4)

For the flow of a viscous incompressible fluid between two parallel plates, one in uniform motion and the other at rest, it is assumed that
\[ \frac{\partial p}{\partial x} = 0, \] since the flow is due to shear only,

and
\[ \frac{\partial p}{\partial y} = 0, \] since the pressure variation at right angles to the plate is neglected for small suction.

Further
\[ \begin{aligned} w &= 0 \\ \frac{\partial}{\partial z} &= 0 \end{aligned} \]

for two-dimensional flow.

Hence Equation (3) identically vanishes and Equations (1), (2) and (4) become
\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \] (5)

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \] (6)

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \] (7)

If the axis of \( x \) is taken along the plate at rest and \( y \) is measured at right angles to it, the boundary conditions are:
\[ \begin{aligned} y &= 0, & v &= -v_0, & u &= 0 \\ y &= y_0, & v &= 0, & u &= U \end{aligned} \] (8)
where $y_0$ is the distance between the two plates, $v_0$ is the uniform suction at the plate at rest and $U$ is the velocity of the plate in motion.

Because of uniform suction $\partial v/\partial x = 0$ and the equation of motion in $y$ direction becomes

$$v \frac{\partial v}{\partial y} = \nu \frac{\partial^2 v}{\partial y^2}$$

(9)

with the boundary conditions

$$v = -v_0 \quad \text{when} \quad y = 0$$

$$v = 0 \quad \text{when} \quad y = y_0.$$ 

If the suction velocity is assumed to be very small, the term on the left-hand side of equation (9) is very small and the right-hand side must therefore be similarly small and approximately equal to zero. Then,

$$\frac{\partial^2 v}{\partial y^2} = 0.$$ 

(11)

The solution of Equation (11) with conditions (10) is

$$v = v_0 \left( \frac{y}{y_0} - 1 \right)$$

(12)

or

$$\frac{v}{v_0} = 1 - \eta,$$

where

$$\eta = \frac{y}{y_0}.$$ 

The distribution of transverse velocity has been shown in Fig. 1.

From Equation (7),

$$\frac{\partial u}{\partial x} = - \frac{\partial v}{\partial y} = - \frac{v_0}{y_0}.$$ 

(13)

Hence

$$\frac{\partial^2 u}{\partial x^2} = 0.$$ 

(14)
Substituting from equations (12), (13) and (14) into Equation (5) we get
\[ \nu \frac{\partial^2 u}{\partial y^2} - v_0 \left( \frac{v}{O} - 1 \right) \frac{\partial u}{\partial y} + \frac{v_0}{y_0} u = 0 \]

or
\[ \frac{\partial^2 u}{\partial \eta^2} - \sigma (\eta - 1) \frac{\partial u}{\partial \eta} + \sigma u = 0 \]  \hspace{1cm} (15)

where
\[ \sigma = \frac{v_0 y_0}{\nu} \]

with the conditions
\[ u = 0 \quad \text{when} \quad \eta = 0 \]

and
\[ u = U \quad \text{when} \quad \eta = 1. \] \hspace{1cm} (16)

Let
\[ \xi = (1 - \eta) \sqrt{\sigma}. \]
Then Equation (15) changes to
\[ \frac{\partial^2 u}{\partial \xi^2} - \xi \frac{\partial u}{\partial \xi} + u = 0 \] (17)
and the conditions (16) become
\[ u = 0 \text{ when } \xi = \sqrt{\sigma} \]
and
\[ u = U \text{ when } \xi = 0. \] (18)

The series solution of Equation (17) is
\[ u = a_1 \xi - a_0 \sum_{n=1}^{\infty} \frac{1}{2^n (2n - 1)} \sqrt{n} \xi^{2n}. \] (19)
Applying the conditions (18), we have
\[ a_0 = U \]
and
\[ a_1 = \frac{U}{\sqrt{\sigma}} \sum_{n=1}^{\infty} \frac{1}{2^n (2n - 1)} \sqrt{n} \sigma^n. \]
Hence,
\[ \frac{u}{U} = (1 - \eta) (S_0 - S) \] (20)
where
\[ S_0 = \sum_{n=1}^{\infty} \frac{1}{2^n (2n - 1)} \sqrt{n} \sigma^n, \]
\[ S = \sum_{n=1}^{\infty} \frac{1}{2^n (2n - 1)} \sqrt{n} \sigma^n (1 - \eta)^{2n-1}. \]
For
\[ \sigma = 0, \quad \frac{u}{U} = \eta = \frac{y}{y_0} \]
as in the case of plane Couette motion without suction. The longitudinal velocity has been calculated for various values of $\eta$ and $\sigma$ and the results of calculation are shown in Table I and have been plotted in Fig. 1.
TABLE I
Values of $u/U = (1 - \eta) (S_0 - S)$ for various values of $\eta$ and $\sigma$

<table>
<thead>
<tr>
<th>$\eta \rightarrow$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma \downarrow$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.000</td>
<td>0.100</td>
<td>0.200</td>
<td>0.300</td>
<td>0.400</td>
<td>0.500</td>
<td>0.600</td>
<td>0.700</td>
<td>0.800</td>
<td>0.900</td>
<td>1.000</td>
</tr>
<tr>
<td>0.2</td>
<td>0.000</td>
<td>0.109</td>
<td>0.217</td>
<td>0.322</td>
<td>0.425</td>
<td>0.526</td>
<td>0.625</td>
<td>0.722</td>
<td>0.816</td>
<td>0.909</td>
<td>1.000</td>
</tr>
<tr>
<td>0.4</td>
<td>0.000</td>
<td>0.120</td>
<td>0.235</td>
<td>0.345</td>
<td>0.444</td>
<td>0.553</td>
<td>0.651</td>
<td>0.744</td>
<td>0.833</td>
<td>0.919</td>
<td>1.000</td>
</tr>
<tr>
<td>0.6</td>
<td>0.000</td>
<td>0.133</td>
<td>0.254</td>
<td>0.371</td>
<td>0.482</td>
<td>0.592</td>
<td>0.677</td>
<td>0.768</td>
<td>0.851</td>
<td>0.929</td>
<td>1.000</td>
</tr>
<tr>
<td>0.8</td>
<td>0.000</td>
<td>0.143</td>
<td>0.276</td>
<td>0.398</td>
<td>0.509</td>
<td>0.613</td>
<td>0.707</td>
<td>0.793</td>
<td>0.870</td>
<td>0.939</td>
<td>1.000</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000</td>
<td>0.157</td>
<td>0.299</td>
<td>0.427</td>
<td>0.542</td>
<td>0.645</td>
<td>0.737</td>
<td>0.819</td>
<td>0.890</td>
<td>0.950</td>
<td>1.000</td>
</tr>
</tbody>
</table>

The shearing stress at the wall is

$$\tau_0 = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}$$

$$= -\frac{\mu U}{y_0} \left[ \frac{\partial}{\partial (1 - \eta)} \{(1 - \eta)(S_0 - S)\} \right]_{1-\eta=1}$$

i.e.,

$$\frac{\tau_0 y_0}{\mu U} = \sum_{o}^{\infty} \frac{1}{2^n \sigma^n}.$$  \hspace{1cm} (21)

The discharge per unit breadth of the plates is

$$Q = \int_{0}^{y_0} u dy = U y_0 \int_{0}^{1} (S_0 - S) Z dZ$$

where $Z = 1 - \eta$ and $u/U$ is a function of $Z$.

Hence

$$Q = U y_0 \int_{0}^{1} \left[ S_0 Z - \sum_{o}^{\infty} \frac{1}{2^n (2n - 1) \sigma^n Z^{2n}} \right] dZ$$

$$= U y_0 \left[ \frac{S_0}{2} - \sum_{o}^{\infty} \frac{1}{2n (2n - 1) (2n + 1) \sigma^n} \right].$$  \hspace{1cm} (22)
The discharge for solid plates \( (\sigma = 0) \) is

\[
Q_0 = \frac{U \gamma_0}{2}.
\]

The discharge coefficient is

\[
C_Q = \frac{Q}{Q_0} = \left[ S_0 - \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{1}{(2n-1)(2n+1)} \frac{1}{n} \sigma^n \right]
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{2n(2n+1)} \frac{1}{n} \sigma^n.
\]

The shearing stress at the stationary plate and the flow coefficient have been calculated for various values of \( \sigma \) and the results of calculations have been shown in Table II and plotted in Fig. 2.

**Table II**

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( S_0 )</th>
<th>( \tau_0 \gamma_0 / \mu U )</th>
<th>( C_Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>-1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>0.100</td>
<td>-0.950</td>
<td>1.051</td>
<td>1.017</td>
</tr>
<tr>
<td>0.200</td>
<td>-0.898</td>
<td>1.105</td>
<td>1.034</td>
</tr>
<tr>
<td>0.300</td>
<td>-0.846</td>
<td>1.162</td>
<td>1.052</td>
</tr>
<tr>
<td>0.400</td>
<td>-0.793</td>
<td>1.221</td>
<td>1.071</td>
</tr>
<tr>
<td>0.500</td>
<td>-0.739</td>
<td>1.284</td>
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<tr>
<td>0.600</td>
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<td>1.110</td>
</tr>
<tr>
<td>0.700</td>
<td>-0.628</td>
<td>1.419</td>
<td>1.130</td>
</tr>
<tr>
<td>0.800</td>
<td>-0.571</td>
<td>1.492</td>
<td>1.151</td>
</tr>
<tr>
<td>0.900</td>
<td>-0.513</td>
<td>1.568</td>
<td>1.173</td>
</tr>
<tr>
<td>1.000</td>
<td>-0.454</td>
<td>1.649</td>
<td>1.195</td>
</tr>
</tbody>
</table>
The shear flow between two walls, one in uniform motion and the other at rest, with uniform suction at the stationary wall.

The volume of fluid removed through suction is given by

\[ Q' = -v_0 \cdot A = -\frac{v_0}{U} \cdot AU \]

where \( A \) is the wetted area of the stationary plate.

**CONCLUSIONS**

The following results have been obtained for the shear flow between two plates, with uniform suction at the stationary plate.

(a) The longitudinal velocity distribution

\[ \frac{u}{U} = (1 - \eta) (S_0 - S), \]
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(b) Transverse velocity distribution

\[ \frac{v}{v_0} = 1 - \eta. \]

(c) Shearing stress at the wall

\[ \frac{\tau_0 y_0}{\mu U} = \sum_{n=0}^{\infty} \frac{1}{2^n \sqrt{n}} \sigma^n. \]

(d) Flow coefficient

\[ C_0 = \sum_{n=0}^{\infty} \frac{1}{2^n (2n + 1) \sqrt{n}} \sigma^n. \]

As a consequence of small suction at the stationary wall, a linear transverse velocity is superimposed over the longitudinal velocity due to shear. The longitudinal velocity distribution between the two plates becomes parabolic for small velocities of suction. The velocity \( u \) and the flow coefficient \( C_0 \) increase as \( \sigma \) increases. The longitudinal velocity and the volume rate of flow decrease along the length of the wall.

Proceeding to the limit \( \sigma = 0 \), we have

\[ \frac{u}{U} = \frac{y}{y_0}, \]

\[ v = 0, \]

\[ \frac{\tau_0 y_0}{\mu U} = 1 \]

and

\[ C_0 = 1 \]

as is the plane Couette flow between two plates. Since the volume flow is decreasing all along the wall, the Reynolds number is similarly decreasing and if the flow is laminar at the leading edge, it will remain so. The ever-decreasing longitudinal velocity and the uniform suction velocity will at some point lead to \( v_0 \) being of comparable magnitude to \( u \) and the solution may cease to hold good under these conditions.

**Summary**

In this paper an attempt has been made to find the solution of the Navier-Stokes equations for the flow of a viscous incompressible fluid between two
plates, one at rest and the other in uniform motion, with small uniform suction at the stationary plate. A solution has been obtained under the assumption that the pressure between the two plates is uniform. It has been shown that due to suction a linear transverse velocity is superimposed over the longitudinal velocity. With suction, the longitudinal velocity distribution between the plates becomes parabolic and decreases along the length of the plate.

The longitudinal velocity, the shearing stress at the stationary plate, and the volume rate of flow increase with \( \sigma = \frac{v_0 y_0}{v} \), the suction parameter defined with reference to the suction velocity, and the distance between the two plates. For \( \sigma = 0 \) the results transform to the known results for plane Couette flow without suction.

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**REFERENCES**