GENERATING FUNCTIONS FOR GEGENBAUER AND LEGENDRE POLYNOMIALS

II. Some Generalisations

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THE aim of this note is to generalise some of the results concerning the bilateral generating functions involving Legendre's and Gegenbauer's polynomials.

§ 1

The sums of infinite series like

$$\Sigma P_n(x) P_n(y) t^n$$

and

$$\sum \frac{C_n^{a+\frac{1}{2}}(x) C_n^{a+\frac{1}{2}}(y) t^n . n!}{(1+2a)_n}$$

have been evaluated already (*cf.* Geronimus[1], Watson[2], Ossicini[3]).

The two following generalised forms of these are given in this note:

$$\sum \frac{P^b_{a+n}(x) C_n^{\beta+\frac{1}{2}}(y) t^n (1-b+a)_n}{(1+2\beta)_n}$$

$$= \frac{2^b (1-x^2)^{-\frac{1}{2}b}}{\Gamma(1-b)(x-ty)^{(1+a-b)}} . F_4\left[\frac{1+a-b}{2}, 1+\frac{a-b}{2},\right.$$

$$\left. 1+\beta, 1-b; \frac{-t^2(1-y^2)}{(x-ty)^2}, \frac{-(1-x^2)}{(x-ty)^2}\right] \quad (1)$$

$$\sum \frac{C_n^{a+\frac{1}{2}}(x) C_n^{\beta+\frac{1}{2}}(y) t^n n!}{(1+2\beta)_n}$$

$$= (x-ty)^{-(1+2a)} F_4\left[\frac{1}{2}+a, 1+a, 1+\beta, 1+a;\right.$$

$$\left.\frac{-t^2(1-y^2)}{(x-ty)^2}, \frac{-(1-x^2)}{(x-ty)^2}\right]. \quad (2)$$

[$F_4(a, \beta, \gamma, \delta; x, y)$ is the hypergeometric function of two variables].

We mention, in passing, that the sums of the following series [which are particular cases of (1) and (2)] can be expressed in terms of Gaussian hypergeometric functions and, for the sake of brevity, the explicit forms are not presented here:

$$\sum \frac{P^b_{a+n}(x)\, C_n^{a+1}(y)\, t^n (1-b+a)_n}{(2+2a)_n}, \tag{3}$$

$$\sum \frac{C_n^{a+\frac{1}{2}}(x)\, C_n^{a+1}(y)\, t^n\, n!}{(2+2a)_n} \tag{4}$$

and

$$\sum \frac{C_n^{a+\frac{1}{2}}(x)\, C_n^{a}(y)\, t^n\, n!}{(2a)_n}. \tag{5}$$

§ 2

To prove (1), we make use of the identity

$$\sum \frac{P^b_{a+n}(x)\, \psi_n^*(y)\, t^n (1-b+a)_n}{n!}$$

$$= \frac{\rho^{-a-1} \sum \psi_n^0\, P^b_{a+n}\left(\frac{x-t}{\rho}\right)\left(\frac{yt}{\rho}\right)^n (1-b+a)_n}{n!} \tag{6}$$

wherein

$$\rho = (1 - 2xt + t^2)^{\frac{1}{2}}$$

and

$\psi_n^*(y)$ is the polynomial defined by

$$\psi_n^*(y) = \sum_r \binom{n}{r} \psi_r^\circ\, y^r. \tag{7}$$

The identity (6) has been proved by the author to be true in an earlier communication.[4] Assuming $\psi_n^*(y)$ as

$$\frac{n!\,(1+y^2)^{n/2}}{(1+2a)_n}\, C_n^{a+\frac{1}{2}}\left(\frac{1}{\sqrt{1+y^2}}\right)$$

with

$$\psi_{2r}^\circ = \frac{(-1)^r (\frac{1}{2})_r}{(1+a)_r}$$

and

$$\psi^{\circ}_{2r+1} = 0,$$

and substituting in (6), we obtain,

$$\sum \frac{P^{b}_{a+n}(x)\, C_n^{\beta+\frac{1}{2}}\left(\frac{1}{\sqrt{1+y^2}}\right)(t\sqrt{1+y^2})^n (1-b+a)_n}{(1+2\beta)_n}$$

$$= \frac{\rho^{-a-1}\sum P^{b}_{a+2n}\left(\frac{x-t}{\rho}\right)\left(-\frac{y^2t^2}{4\rho^2}\right)^n (1-b+a)_{2n}}{(1+\beta)_n \cdot n!}. \tag{8}$$

Since,

$$P_\nu^\mu(x) = \frac{2^\mu (1-x^2)^{-\frac{1}{2}\mu}}{\Gamma(1-\mu)} F\left(\frac{1+\nu-\mu}{2}, -\frac{\nu-\mu}{2}; 1-\mu; 1-x^2\right), \tag{9}$$

$$P^{b}_{a+2n}\left(\frac{x-t}{\rho}\right)$$

$$= \frac{2^b \rho^b (1-x^2)^{-\frac{1}{2}b}}{\Gamma(1-b)}\left(\frac{x-t}{\rho}\right)^{-(1+a+2n-b)}$$

$$\times F\left[n+\frac{1}{2}(1+a-b),\ n+1+\frac{1}{2}(a-b);\ 1-b;\ 1-\frac{\rho^2}{(x-t)^2}\right]. \tag{10}$$

Substituting (10) in (8), it is seen that

$$\sum \frac{P^{b}_{a+n}(x)\, C_n^{a+\frac{1}{2}}(y)\, t^n (1-b+a)_n}{(1+2\beta)_n}$$

$$= \frac{2^b (1-x^2)^{-\frac{1}{2}b}}{\Gamma(1-b)(x-ty)^{1+a-b}} F_4\left[\frac{1}{2}(1+a-b),\ 1+\frac{1}{2}(a-b),\ 1+\beta,\ 1-b;\ \frac{-t^2(1-y^2)}{(x-ty)^2},\ \frac{-(1-x^2)}{(x-ty)^2}\right] \tag{11}$$

which is (1). With $b = -a$, (1) becomes, after rearrangements,

$$\sum \frac{C_n^{a+\frac{1}{2}}(x)\, C_n^{\beta+\frac{1}{2}}(y)\, t^n (1+2a)_n}{(1+2\beta)_n}$$

$$= (x - ty)^{-(1+2\alpha)} \mathrm{F}_4 \left[\frac{1}{2} + \alpha,\ 1 + \alpha,\ 1 + \beta,\ 1 + \alpha;\ \frac{-t^2(1-y^2)}{(x-ty)^2},\ \frac{-(1-x^2)}{(x-ty)^2}\right] \tag{12}$$

which is (2).

§ 3

In particular, let us take $\alpha = \beta$ in (12).

Then,

$$\Sigma \mathrm{C}_n^{\alpha+\frac{1}{2}}(x)\, \mathrm{C}_n^{\alpha+\frac{1}{2}}(y)\, t^n$$

$$= (x - ty)^{-(1+2\alpha)} \mathrm{F}_4 \left[\frac{1}{2} + \alpha,\ 1 + \alpha,\ 1 + \alpha,\ 1 + \alpha;\ \frac{-t^2(1-y^2)}{(x-ty)^2},\ \frac{-(1-x^2)}{(x-ty)^2}\right]. \tag{13}$$

But,

$$\mathrm{F}_4\left[\alpha, \beta, \beta, \beta;\ \frac{-\mathrm{X}}{(1-\mathrm{X})(1-\mathrm{Y})},\ \frac{-\mathrm{Y}}{(1-\mathrm{X})(1-\mathrm{Y})}\right]$$

$$= (1-\mathrm{X})^{\alpha}(1-\mathrm{Y})^{\alpha}\, \mathrm{F}(\alpha, 1+\alpha-\beta;\ \beta;\ \mathrm{XY}) \tag{14}$$

(*cf.* Erdelyi[5])

Writing

$$\frac{-t^2(1-y^2)}{(x-ty)^2} = \frac{-\mathrm{X}}{(1-\mathrm{X})(1-\mathrm{Y})}$$

and

$$\frac{-(1-x^2)}{(x-ty)^2} = \frac{-\mathrm{Y}}{(1-\mathrm{X})(1-\mathrm{Y})},$$

it is seen that,

$$\sqrt{\mathrm{XY}} = \frac{\sqrt{1+t^2-2t\cos(\alpha+\beta)} - \sqrt{1+t^2-2t\cos(\alpha+\beta)}}{\sqrt{1+t^2-2t\cos(\alpha+\beta)} + \sqrt{1+t^2-2t\cos(\alpha-\beta)}}$$

and

$$(1 - X)(1 - Y)$$

$$= \frac{4(\cos\alpha - t\cos\beta)^2}{\sqrt{1 + t^2 - 2t\cos(\alpha + \beta)} + \sqrt{1 + t^2 - 2t\cos(\alpha - \beta)}}, \quad (15)$$

where, for convenience, we write $x = \cos\alpha$ and $y = \cos\beta$. Hence, by using (13), (14) and (15), one can easily deduce that,

$$\Sigma C_n^{a+\frac{1}{2}}(\cos\alpha)\, C_n^{a+\frac{1}{2}}(\cos\beta)\, t^n$$

$$= \tfrac{1}{2}\{\sqrt{1 + t^2 - 2t\cos(\alpha + \beta)}$$

$$+ \sqrt{1 + t^2 - 2t\cos(\alpha - \beta)}\}^{-(1+2a)}$$

$$\times F(\tfrac{1}{2} + a, \tfrac{1}{2}; 1 + a; k^2) \quad (16)$$

with

$$k = \frac{\sqrt{1 + t^2 - 2t\cos(\alpha + \beta)} - \sqrt{1 + t^2 - 2t\cos(\alpha - \beta)}}{\sqrt{1 + t^2 - 2t\cos(\alpha + \beta)} + \sqrt{1 + t^2 - 2t\cos(\alpha - \beta)}}.$$

With $a = 0$, in (16), the well-known result, *viz.*,

$$\Sigma P_n(\cos\alpha)\, P_n(\cos\beta)\, t^n$$

$$= \tfrac{1}{2}\{\sqrt{1 + t^2 - 2t\cos(\alpha + \beta)}$$

$$+ \sqrt{1 + t^2 - 2t\cos(\alpha - \beta)}\}^{-1} \frac{2K(k)}{\pi}$$

where the elliptic function

$$K(k) = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2)$$

[*cf.* Refs. (1), (2) and (3)] is obtained.

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