SLOW ROTATION OF A SPHERE IN A NON-NEWTONIAN FLUID WITH OR WITHOUT SUCTION OR INJECTION

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Abstract

The flow generated by the rotation of a sphere in an infinitely extending fluid has recently been studied by Goldshtik. The corresponding problem for non-Newtonian Reiner-Rivlin fluids has been studied by Datta. Bhatnagar and Rajeswari have studied the secondary flow between two concentric spheres rotating about an axis in the non-Newtonian fluids. This last investigation was further generalised by Rajeswari to include the effects of small radial suction or injection.

In Part A of the present investigation, we have studied the secondary flow generated by the slow rotation of a single sphere in non-Newtonian fluid obeying the Rivlin-Ericksen constitutive equation. In Part B, the effects of small suction or injection have been studied which is applied in an arbitrary direction at the surface of the sphere.

In the absence of suction or injection, the secondary flow for small values of the visco-elastic parameter is similar to that of Newtonian fluids with inclusion of inertia terms in the Oseen approximation. If this parameter exceeds $K_c = 18R/219$, where R is the Reynolds number, the breaking of the flow field takes place into two domains, in one of which the stream lines form closed loops. For still higher values of this parameter, the complete reversal of the sense of the flow takes place.

When suction or injection is included, the breaking of the flow persists under certain condition investigated in this paper. When this condition is broken, the breaking of the flow is obliterated.

1. Introduction

The flow generated by the rotation of a sphere in an infinitely extending fluid has recently been studied by Goldshtik[1]. Datta[2] has studied the corresponding problem for the visco-inelastic fluids obeying the Reiner-Rivlin constitutive equation

$$T = -pI + \phi_1 A + \phi_3 A^2, \tag{1.1}$$

where ϕ_1 and ϕ_3 are respectively the constant coefficients of viscosity and cross-viscosity; T is the stress tensor, and

$$A_{ij} = u_{i,j} + u_{j,i} \tag{1.2}$$

is the rate of strain tensor. The main aim of these papers was to investigate the secondary currents of small intensity arising in a meridional section due to the pressure difference occurring between the pole and the equator. Bhatnagar and Rajeswari[3] have also recently studied the secondary flow between two concentric spheres rotating about an axis in the non-Newtonian fluids. The last investigation has been generalised by Rajeswari[4] to include the effects of small radial suction or injection.

In Part A of the present paper, we have studied the secondary flow generated by the slow rotation of a single sphere in a visco-elastic non-Newtonian fluid obeying the Rivlin-Ericksen constitutive equation

$$T = -pI + \phi_1 A + \phi_2 B, \tag{1.3}$$

where ϕ_2 is the constant coefficient of visco-elasticity and

$$B_{ij} = a_{i,j} + a_{j,i} + u_{m,i}u^m{}_{,j} + u_{m,j}u^m{}_{,i} \tag{1.4}$$

is the acceleration gradient tensor and a_i, the acceleration.

In Part B, we have studied the effects of small suction or injection for both visco-inelastic non-Newtonian fluid (1.1) and visco-elastic non-Newtonian fluid (1.3). We note that in the present investigation we have taken suction or injection in an arbitrary direction at the surface of the sphere as against the radial suction or injection in (4).

We find that in the case of the visco-elastic fluid with no suction or injection, the secondary flow is similar to that of the Newtonian fluids with the inclusion of inertia terms in the Oseen approximation if the visco-elastic parameter $K = \phi_2 \Omega / \phi_1$ is small. For larger values of K, the breaking of the flow in two parts takes place in one of which the stream lines for the secondary flow are closed loops. For still higher values of K, reversal in the sense of the flow takes place. Datta[2] has predicted the reversal of the secondary flow for the Reiner-Rivlin fluids, though he has not studied the breaking of the flows. However, we have checked from his equations that similar breaking of the secondary flow takes place in his case also. We may also mention that in the case of two sphere problem studied in (3) these features exist.

When we include small suction or injection, the breaking of the flow persists for small suctions under certain condition which are investigated here. However, when this condition is broken the breaking is obliterated. For particluar suction laws, we have drawn stream lines for the secondary flow for the value one of the Reynolds number $a^2\Omega\rho/\phi_1$, a being the radius of the sphere and $K = 0{\cdot}025$. We have not drawn any graphs for the visco-inelastic fluid as the stream lines in general follow the same pattern as for visco-elastic fluid.

PART A

2. Let us consider the sphere of radius 'a' rotating about a certain axis with constant angular velocity Ω. We shall be working through spherical polar co-ordinates with origin at the centre of the sphere and the polar angle θ being measured from the axis of rotation. Let the half line $\theta = 0$ coincide with the positive direction of the axis of rotation of the sphere. Further we consider the motion to the steady and axially symmetric so that $\partial/\partial t = 0$ and $\partial/\partial\phi = 0$.

Let the components of velocity in the increasing directions of r, θ and ϕ be U, V and W respectively. We have also assumed that $U \ll W$ and $V \ll W$, *i.e.*, we neglect the products and powers of U and V according to the Oseen's scheme. We shall use all the variables rendered dimensionless using 'a' as the characteristic length and $a\Omega$ as the standard velocity. In this scheme, the hydrostatic pressure will then be given by $\phi_1\Omega p$.

The equations of continuity and momentum are:

$$\frac{\partial U}{\partial r} + \frac{2U}{r} + \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{V}{r}\cot\theta = 0 \tag{2.1}$$

$$RA = -\frac{\partial p}{\partial r} + \left(\triangle U - \frac{2U}{r^2} - \frac{2}{r^2}\frac{\partial V}{\partial \theta} - \frac{2V\cot\theta}{r^2}\right) + KK_1 \tag{2.2}$$

$$RB = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \left(\triangle V + \frac{2}{r^2}\frac{\partial U}{\partial \theta} - \frac{V}{r^2\sin^2\theta}\right) + KK_2 \tag{2.3}$$

$$RC = \left(\triangle W - \frac{W}{r^2\sin^2\theta}\right) + KK_3 \tag{2.4}$$

where

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} \tag{2.5}$$

$$R = \frac{a^2\Omega\rho}{\phi_1}, \quad K = \frac{\phi_2\Omega}{\phi_1} \tag{2.6}$$

are the Reynolds number and the dimensionless parameter denoting the relative magnitude of ϕ_2 to ϕ_1 and

$$A = U\frac{\partial U}{\partial r} + \frac{V}{r}\frac{\partial U}{\partial \theta} - \frac{V^2 + W^2}{r} \tag{2.7}$$

$$B = U\frac{\partial V}{\partial r} + \frac{V}{r}\frac{\partial V}{\partial \theta} + \frac{UV}{r} - \frac{W^2}{r}\cot\theta \tag{2.8}$$

$$C = U\frac{\partial W}{\partial r} + \frac{V}{r}\frac{\partial W}{\partial \theta} + \frac{UW}{r} + \frac{VW}{r}\cot\theta \tag{2.9}$$

K_1, K_2, K_3 are the same as given in (5) but for completeness we are recording them in the appendix.

The boundary conditions of the problem are

$$\left.\begin{aligned} &U = V = 0 \quad \text{at} \quad r = 1 \\ &W = \sin\theta \quad \text{at} \quad r = 1 \\ &U = V = W = p = 0 \quad \text{as} \quad r \to \infty \end{aligned}\right\}. \tag{2.10}$$

The primary motion, *i.e.*, the rectilinear motion with the neglect of inertia terms and visco-elasticity is given by

$$U = V = 0, \quad W = \frac{\sin\theta}{r^2}. \tag{2.11}$$

The equation (2.1) permits us to introduce the stream function ψ defined by

$$U = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \tag{2.12}$$

$$V = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{2.13}$$

so that the continuity equation is identically satisfied.

We further try to find out the function ψ in the form

$$\psi = f(r)\sin\theta\sin 2\theta. \tag{2.14}$$

Thus (2.12) and (2.13) give

$$U = -\frac{2f}{r^2}(\cos^2\theta + \cos 2\theta) \tag{2.15}$$

$$V = +\frac{f'}{r}\sin 2\theta \tag{2.16}$$

where dash denotes differentiation with respect to r.

The secondary motion is then given by

$$\frac{\partial p}{\partial r} = \frac{\mathrm{R}\sin^2\theta}{r^5} + 2\left(\cos^2\theta + \cos 2\theta\right)\left(\frac{6f}{r^4} - \frac{f''}{r^2}\right) - \frac{73\sin^2\theta}{r^7}\mathrm{K} \tag{2.17}$$

$$\frac{\partial p}{\partial \theta} = \frac{\mathrm{R}\sin\theta\cos\theta}{r^4} + 2\sin\theta\cos\theta\left\{f''' - \frac{6}{r^2}f' + \frac{12f}{r^3}\right\}. \tag{2.18}$$

Integrating (2.17) with respect to r we get

$$p = -\frac{\mathrm{R}\sin^2\theta}{4r^4} + 2\left(\cos^2\theta + \cos 2\theta\right)\left[-\left(\frac{f}{r^2}\right)' - \frac{4f}{r^3}\right]$$

$$+\frac{73\mathrm{K}\sin^2\theta}{6r^6} + \mathrm{F}(\theta). \tag{2.19}$$

Since $p = 0$ as $r \rightarrow \infty$, $\mathrm{F}(\theta) = 0$.

Substituting in (2.18) for $\partial p/\partial\theta$ from (2.19) we get the equation determining f in the form

$$r^3 f''' - 12\, r f' = -\frac{3}{4r}\mathrm{R} + \frac{73}{6}\mathrm{K}\frac{1}{r^3} \tag{2.20}$$

which gives

$$f = \mathrm{C}_1 + \mathrm{C}_2 r^5 + \frac{\mathrm{C}_3}{r^2} - \frac{1}{8r}\mathrm{R} - \frac{73\mathrm{K}}{144}\frac{1}{r^3}. \tag{2.21}$$

Now $f/r^2 \rightarrow 0$ as $r \rightarrow \infty$ and therefore $\mathrm{C}_2 = 0$.

Also at $r = 1$, $\mathrm{U} = \mathrm{V} = 0$, *viz.*, $f(1) = f'(1) = 0$, which give

$$f = \frac{(1-r)^2}{16r^2}\left[\mathrm{R} - \frac{73}{18}\mathrm{K}\frac{r+2}{r}\right] \tag{2.22}$$

and hence from (2.14)

$$\psi = \frac{(1-r)^2}{8r^2}\left[\mathrm{R} - \frac{73}{18}\mathrm{K}\frac{r+2}{r}\right]\sin^2\theta\cos\theta \tag{2.23}$$

3. DISCUSSION OF THE RESULTS

From (2.23) we have for $0 \leq \theta \leq \pi/2$,

$\psi \gtrless 0$ according as

$$r \gtrless \frac{146\text{K}}{18\text{R} - 73\text{K}}. \tag{3.1}$$

Also $\psi = 0$, where

$$\frac{\text{R}}{\text{K}} = \frac{73}{18}\left(1 + \frac{2}{r}\right). \tag{3.2}$$

Therefore the circle $r = r_0 = 146\,\text{K}/(18\,\text{R} - 73\,\text{K})$ in the meridian plane divides the flow field into two domains: In the inner domain the stream lines correspond to negative values of ψ, while in the outer domain they correspond to positive values of ψ. We may check that the radial velocity U vanishes on the dividing stream line $\psi = 0$. The inequality (3.1) is extremely important from the point of breaking of the flow. If $r_0 < 1$, there cannot be any breaking and this fact gives us the critical value for $\text{K}_\text{C} = 18\,\text{R}/219$ such that when $\text{K} < \text{K}_\text{C}$ there is no breaking of the flow.

In Fig. 1, we have plotted the stream lines $\psi = 0{\cdot}0003$ for $\text{K} = 0$ (Newtonian fluid) and $\text{K} = 0{\cdot}005$ ($< \text{K}_\text{C}$) for a fixed value of the Reynolds number $\text{R} = 1$. We notice that for this value of K there is no breaking of the flow field. The stream lines for the visco-elastic and Newtonian fluid are similar but they are displaced away from the sphere when compared with the corresponding stream lines for the Newtonian fluid. The lines $\theta = 0$ and $\theta = \pi/2$ behave as the asymptotes for the stream lines.

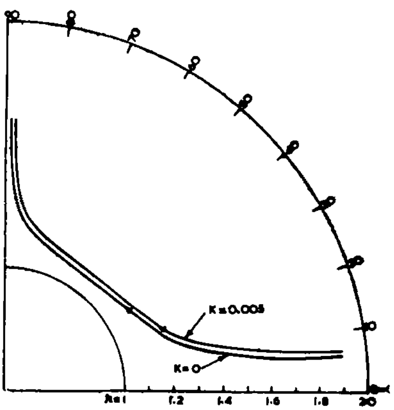

FIG. 1. Comparison of the stream lines $\psi = 0{\cdot}0003$ for Newtonian [K = 0] and visco-elastic fluids. [K = 0·005] for R = 1.

In Fig. 2, we have plotted the stream lines $\psi = 0 \cdot 00005$ and $\psi = 0 \cdot 000057$ for critical value of visco-elastic parameter, *viz.*, $K = K_C = 0 \cdot 025$ for $r_0 = 1 \cdot 36$ and $R = 1$, for which the breaking of the secondary flow takes place into the closed loops within the stream line $\psi = 0$ and the stream line pattern is similar to that of the Newtonian fluids outside $\psi = 0$ stream line.

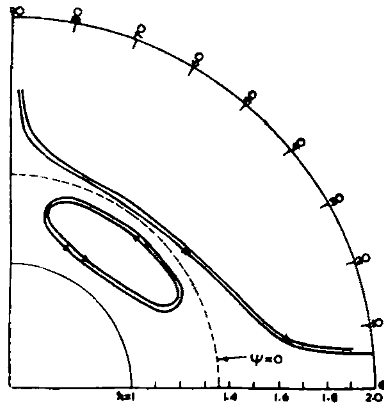

FIG. 2. The secondary flow pattern for a critical value of $K = 0 \cdot 025$, $R = 1$, $\psi = \pm\ 0 \cdot 00005$, $\pm\ 0 \cdot 000057$.

Figure 3 gives the stream ilne $\psi = 0 \cdot 0005$ for a comparatively larger value of $K = 0 \cdot 05$. We notice here that the complete reversal of the sense of the secondary flow has taken place.

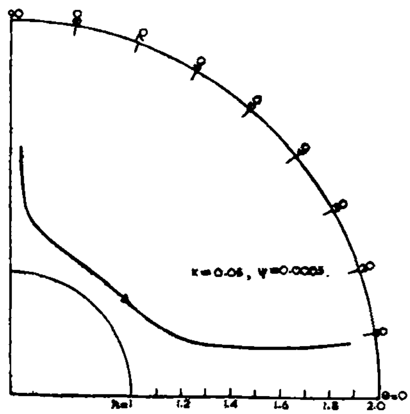

FIG. 3. The secondary flow pattern $\psi = 0 \cdot 0005$ for $K = 0 \cdot 05$ for which reversal of the flow takes place.

In Fig. 4 we have plotted the values of $R/2K$ *versus* r_0, the radius of the dividing stream line with the help of the equation (3.2). With the help of this graph, we can determine the parameter K for a given value of R if we can experimentally find out the position of the $\psi = 0$ stream line. Thus the present investigation adds one more method of determining K experimentally to those already been suggested by Bhatnagar and Rajeswari.[6,3]

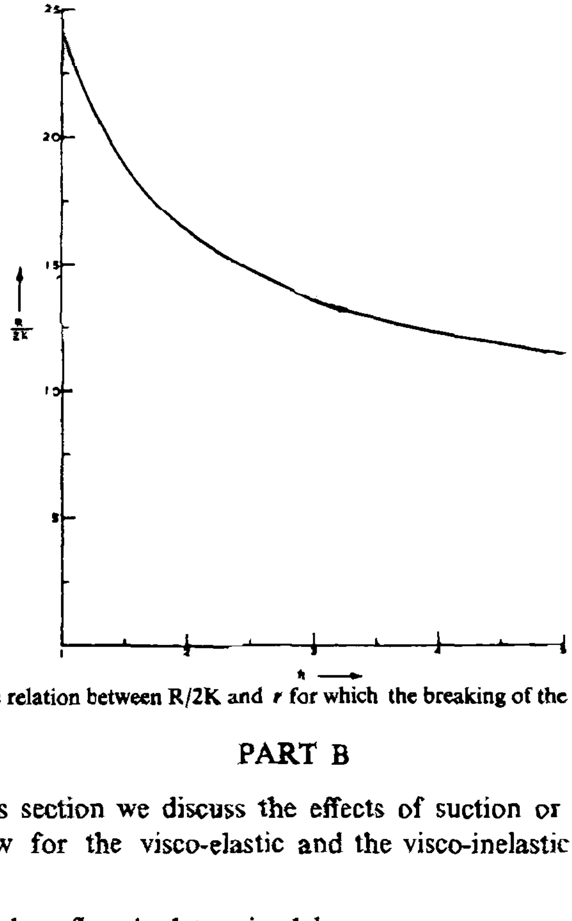

FIG. 4. The relation between R/2K and r for which the breaking of the flow takes place.

PART B

4. In this section we discuss the effects of suction or injection on the secondary flow for the visco-elastic and the visco-inelastic non-Newtonian fluids.

The secondary flow is determined by

$$\frac{\partial p}{\partial r} = \frac{\mathrm{R}\sin^2\theta}{r^5} + \left[-\frac{1}{r^2\sin\theta}\frac{\partial^3\psi}{\partial r\partial\theta} - \frac{1}{r^4\sin\theta}\frac{\partial^3\psi}{\partial\theta^3} + \frac{\cos\theta}{r^4\sin^2\theta}\frac{\partial^2\psi}{\partial\theta^2} - \frac{1}{r^4\sin^3\theta}\frac{\partial\psi}{\partial\theta}\right] - \frac{73\sin^2\theta}{r^7}\mathrm{K} - \frac{45\sin^2\theta}{r^2}\mathrm{S} \tag{4.1}$$

$$\frac{\partial p}{\partial\theta} = \frac{\mathrm{R}\sin\theta\cos\theta}{r^4} + \left[\frac{1}{\sin\theta}\frac{\partial^3\psi}{\partial r^3} - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{1}{r^2\sin\theta}\frac{\partial^3\psi}{\partial r\partial\theta^2} + \frac{2\cos\theta}{r^3\sin^2\theta}\frac{\partial\psi}{\partial\theta} - \frac{2}{r^3\sin\theta}\frac{\partial^2\psi}{\partial\theta^2}\right] - \frac{9\sin\theta\cos\theta}{r^6}\mathrm{S} \tag{4.2}$$

where $S = \phi_2\Omega/\phi_1$, is the dimensionless parameter of cross-viscosity, and the coefficients of S, namely S_1, S_2, S_3 are calculated with the help of expressions given in the Appendix.

Eliminating p between (4.1) and (4.2) we get the equation determining ψ in the form

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial(\cos\theta)^2}\right]\psi$$

$$= \sin^2\theta\cos\theta\left[\frac{6}{r^5}\mathrm{R} - \frac{146}{r^7}\mathrm{K} - \frac{144}{r^7}\mathrm{S}\right]. \tag{4.3}$$

The solution of (4.3) is

$$\psi = \cos\theta\,[a_0 r + b_0 r^2 + d_0 r^3 + \mathrm{C}_0]$$

$$+ \sum_{n=1}^{\infty} \sin\theta\,\mathrm{P}_n{}^1(\mu)\left[\frac{a_n}{r^n} + b_n r^{n+3} + \mathrm{C}_n r^{2-n} + d_n r^{n+1}\right]$$

$$- \sin\theta\,\mathrm{P}_2{}^1(\mu)\left[\frac{\mathrm{R}}{12r} + \frac{\mathrm{S}}{3r^3} + \frac{73}{216}\frac{\mathrm{K}}{r^3}\right] \tag{4.4}$$

where $\mathrm{P}_n{}^1(\mu)$ and $\mathrm{P}_2{}^1(\mu)$ are the usual associated Legendre's Polynomials.

From (2.12) and (2.13) we get

$$\mathrm{U} = \left[\frac{a_0}{r} + b_0 + \frac{\mathrm{C}_0}{r^2} + d_0 r\right] - \sum_{n=1}^{\infty} n(n+1)\,\mathrm{P}_n(\mu)$$

$$\times\,[a_n r^{-(n+2)} + b_n r^{n+1} + \mathrm{C}_n r^{-n} + d_n r^{n-1}]$$

$$+ 2\mathrm{P}_2(\mu)\left[\frac{\mathrm{R}}{4r^3} + \frac{\mathrm{S}}{r^5} + \frac{73}{72}\frac{\mathrm{K}}{r^5}\right] \tag{4.5}$$

$$\mathrm{V} = \cot\theta\left[\frac{a_0}{r} + 2b_0 + 3d_0 r\right] + \sum_{n=1}^{\infty}\mathrm{P}_n{}^1(\mu)$$

$$\times\,[-na_n r^{-(n+2)} + (n+3)\,b_n r^{n+1} + (2-n)\mathrm{C}_n r^{-n} + (n+1)\,d_n r^{n-1}]$$

$$+ \mathrm{P}_2{}^1(\mu)\left[\frac{\mathrm{R}}{12r^3} + \frac{\mathrm{S}}{r^5} + \frac{73}{72}\frac{\mathrm{K}}{r^5}\right]. \tag{4.6}$$

$$\left.\begin{array}{l}\text{Since} \quad U = V = 0 \quad \text{as} \quad r \to \infty, \\ \qquad b_0 = d_0 = 0, \quad b_n = d_n = 0. \\ \text{Further V is finite at } \theta = 0, \text{ which requires } a_0 = 0\end{array}\right\} . \tag{4.7}$$

Thus

$$U = \frac{C_0}{r^2} - \sum_{n=1}^{\infty} n(n+1) P_n(\mu) [a_n r^{-(n+2)} + C_n r^{-n}$$

$$+ 2P_2(\mu) \left[\frac{R}{4r^3} + \frac{S}{r^5} + \frac{73}{72}\frac{K}{r^5}\right] \tag{4.8}$$

$$V = \sum_{n=1}^{\infty} P_n^1(\mu) [- n a_n r^{-(n+2)} + (2-n) C_n r^{-n}]$$

$$+ P_2^1(\mu) \left[\frac{R}{12r^3} + \frac{S}{r^5} + \frac{73}{72}\frac{K}{r^5}\right]. \tag{4.9}$$

We now apply variable suction or injection at the sphere in the radial as well as in the tangential direction in the form:

$$U = \sum_{n=0}^{\infty} \lambda_n P_n(\mu) \quad \text{at} \quad r = 1 \tag{4.10}$$

$$V = \sum_{n=2}^{\infty} \mu_n P_n^1(\mu) \quad \text{at} \quad r = 1 \tag{4.11}$$

where λ_n, μ_n are arbitrary constants. By properly choosing them we can achieve suction or injection in any arbitrary direction relative to a radius vector.

From (4.8)–(4.11) we have,

$$\sum_{n=0}^{\infty} \lambda_n P_n(\mu) = C_0 - \sum_{n=1}^{\infty} n(n+1) P_n(\mu) [a_n + C_n]$$

$$+ 2P_2(\mu) \left[\frac{R}{4} + S + \frac{73}{72} K\right] \tag{4.12}$$

and

$$\sum_{n=2}^{\infty} \mu_n P_n^1(\mu) = \sum_{n=1}^{\infty} P_n^1(\mu) [- n a_n + (2-n) C_n]$$

$$+ P_2^1(\mu) \left[\frac{R}{12} + S + \frac{73}{72} K\right], \tag{4.13}$$

which completely determine all the constants a_n and C_n if the suction or injection parameters λ_n and μ_n are known.

Solving (4.12) and (4.13) we get the following set of constants:

$$C_0 = \lambda_0 \tag{4.14}$$

$$a_1 = C_1 = -\frac{\lambda_1}{4} \tag{4.15}$$

$$\left.\begin{aligned} a_2 &= -\frac{\mu_2}{2} + \frac{R}{24} + \frac{S}{2} + \frac{73}{144}K \\ C_2 &= -\frac{\lambda_2}{6} + \frac{\mu_2}{2} + \frac{R}{24} - \frac{S}{6} - \frac{73}{432}K \end{aligned}\right\} \tag{4.16}$$

and in general for $n \geqslant 3$,

$$\left.\begin{aligned} a_n &= \frac{\lambda n}{2n(n+1)}(n-2) - \frac{1}{2}\mu_n \\ C_n &= \frac{1}{2}\left[\mu_n - \frac{\lambda n}{n+1}\right] \end{aligned}\right\}. \tag{4.17}$$

5. For the numerical computations, we have taken the following particular law of suction for U and V:

$$U = \lambda_0 + \tfrac{1}{2}\lambda_2(3\cos^2\theta - 1), \tag{5.1}$$

$$V = 3\mu_2 \sin\theta \cos\theta. \tag{5.2}$$

Since we have chosen λ_1 to be zero, from (4.15), we get $a_1 = C_1 = 0$. Under these assumptions,

$$\psi = \lambda_0 \cos\theta + 3\sin^2\theta\cos\theta\left[\frac{\mu_2}{2} + \frac{R}{24}\frac{(1-r)^2}{r^2} - \frac{\lambda_2}{6} - \frac{\mu_2}{2r^2} - \frac{(1-r)^2(r+2)}{6r^3}\left(S + \frac{73}{72}K\right)\right]. \tag{5.3}$$

For a visco-elastic fluid ($S = 0$, $K \neq 0$), in the absence of suction or injection, we notice that (5.3) reduces to (2.23) and for a visco-inelastic fluid ($S \neq 0$, $K = 0$) this equation reduces to the equation obtained by Datta.[2]

Case (*i*): *Visco-elastic fluid*, $S = 0,\ K \neq 0$

The stream function ψ for the secondary flow is given by

$$\psi = \cos\theta\left[\lambda_0 + \sin^2\theta\left\{\left(R - \frac{73}{18}K\frac{r+2}{r}\right)\frac{(r-1)^2}{8r^2} - \frac{\lambda_2}{2} - \frac{3\mu_2}{2}\left(\frac{1}{r^2} - 1\right)\right\}\right], \tag{5.4}$$

when the suction or injection at the boundary of the sphere is applied according to the laws (5.1) and (5.2).

In Fig. 5, we have plotted the stream lines in the particular case when $R = 1$, $K = 0\cdot025$, $\lambda_0 = 0\cdot002$, $\lambda_2 = 0\cdot1$ and $\mu_2 = 0\cdot02$. The stream line $\psi = 0$ starts on the sphere and ends on $\theta = \pi/2$ which forms the part of the stream line. This stream line ($\psi = 0$) separates the flow field into two

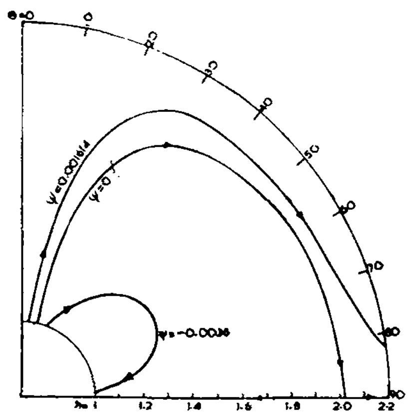

FIG. 5. $R = 1$, $K = 0\cdot025$, $\lambda_0 = 0\cdot002$, $\lambda_2 = 0\cdot1$, $\mu_2 = 0\cdot02$.

domains. In the inner region, the stream lines originate and terminate on the sphere as may be seen from the stream line $\psi = -0\cdot0036$. In the outer region, the stream lines although start on the sphere, extend to infinity as may be seen from the typical stream line $\psi = 0\cdot001{,}614$. We also notice that, if $\lambda_0 < \frac{1}{2}\lambda_2$, the dividing stream line $\psi = 0$ exists in the flow field as the equation

$$\lambda_0 + \sin^2\theta\left\{\left(R - \frac{73}{18}K\frac{r+2}{r}\right)\frac{(r-1)^2}{8r^2} - \frac{\lambda_2}{2} - \frac{3\mu_2}{2}\left(\frac{1}{r^2} - 1\right)\right\} = 0 \tag{5.5}$$

admits values of $r \geq 1$ for values of θ, $\theta_0 \leq \theta < \pi/2$ where $\theta_0 = \sin^{-1}\sqrt{2\lambda_0/\lambda_2}$. But if we adopt values of λ_0, λ_2 which do not satisfy this condition, then $\psi = 0$ is only given by $\cos\theta = 0$, *i.e.*, $\theta = \pi/2$ and then U is always positive (case of injection). In such a case, therefore, $\psi = 0$ cannot divide the flow field into two domains and thus the separation does not take place. Also $\theta = 0$ becomes the stream line for which $\psi = \lambda_0$. Thus ψ varies from $\psi = 0$ at $\theta = \pi/2$ to $\psi = \lambda_0\ (> 0)$ at $\theta = 0$ and in the quadrant $0 < \theta < \pi/2$ the stream lines for which $0 \leq \psi \leq \lambda_0$ start on the sphere and extend to infinity. Thus, if we choose the suction or injection parameters violating the condition, *viz.*,

$$\lambda_0 < \tfrac{1}{2}\lambda_2, \tag{5.6}$$

the flow separation can be avoided. On the other hand, if λ_0 is negative the fluid will be sucked at the boundary in the case of no separation, *i.e.*, the stream lines start from infinity and end on the sphere.

Case (ii): Visco-inelastic fluid, $K = 0,\ S \neq 0$

We notice that the behaviour of the visco-inelastic fluid will be similar to the one described in case (i) as can be easily seen from (5.3) on noting that the coefficients of the terms S and K differ only slightly. We have, therefore, not discussed this case in detail.

6. Conclusions

In the absence of suction or injection, for small values of the visco-elastic parameter K, the secondary flow pattern is similar to the Newtonian fluids with the inclusion of inertial effects according to Oseen's scheme but for the slight displacement of the stream lines. For a critical value of this parameter depending on the Reynolds number and the measure of the rotation of the sphere, the secondary flow is further divided in each quadrant into two regions of which one region is completely closed and the other extends to infinity in contradistinction to the case of Newtonian fluids. In the former region the stream lines form closed loops and in the later region they have the same pattern as in the case of Newtonian fluids. For still higher values of K, the complete reversal of the sense of the secondary flow takes place. We have also shown in §4 that by locating the dividing stream line it is possible to determine K for a given value of R.

In the case of suction or injection, the breaking of the flow is possible if the radial flow paramters satisfy the inequality $\lambda_0 < \frac{1}{2}\lambda_2$. If, however, this condition is violated, the breaking of the flow field can never take place.

APPENDIX

$$K_1 = 2A_{rr} + \frac{1}{r^2} A_{\theta\theta} + \frac{1}{r} B_{r\theta} + \frac{4}{r} A_r + \frac{\cot\theta}{r^2} A_\theta - \frac{3}{r^2} A$$

$$+ \frac{\cot\theta}{r} B_r - \frac{3}{r^2} B_\theta - \frac{3\cot\theta}{r^2} B + 2(u_r^2 + v_r^2 + w_r^2)_r$$

$$+ \frac{2}{r^2} [u_r(u_\theta - v) + v_r(u + v_\theta) + w_\theta w_r]_\theta$$

$$+ \frac{4}{r}(u_r^2 + v_r^2 + w_r^2) - \frac{2}{r^3}[(u_\theta - v)^2 + (u + v_\theta)^2 + w_\theta^2]$$

$$+ \frac{2\cot\theta}{r^2}[u_r(u_\theta - v) + v_r(u + v_\theta) + w_\theta w_r]$$

$$- \frac{2}{r^3\sin^2\theta}(u\sin\theta + v\cos\theta)^2 - \frac{2w^2}{r^3\sin^2\theta},$$

$$K_2 = \frac{1}{r} A_{r\theta} + B_{rr} + \frac{2}{r^2} B_{\theta\theta} + \frac{4}{r^2} A_\theta + \frac{2}{r} B_r + \frac{2\cot\theta}{r^2} B_\theta$$

$$- \frac{2}{r^2\sin^2\theta} B + \frac{2}{r}[u_r(u_\theta - v) + v_r(u + v_\theta) + w_\theta w_r]_r$$

$$+ \frac{2}{r^3}[(u_\theta - v)^2 + (u + v_\theta)^2 + w_\theta^2]_\theta$$

$$+ \frac{4}{r^2}[u_r(u_\theta - v) + v_r(u + v_\theta) + w_\theta w_r]$$

$$+ \frac{2\cot\theta}{r^3}[(u_\theta - v)^2 + (u + v_\theta)^2 + w_\theta^2]$$

$$- \frac{2\cot\theta}{r^3\sin^2\theta}[w^2 + (u\sin\theta + v\cos\theta)^2],$$

$$K_3 = C_{rr} + \frac{1}{r^2} C_{\theta\theta} + \frac{2}{r} C_r + \frac{\cot\theta}{r^2} C_\theta - \frac{1}{r^2\sin^2\theta} C$$

$$- \frac{2}{r}[wu_r + wv_r\cot\theta - w_r(u + v\cot\theta)]_r$$

$$- \frac{2}{r^3}[w(u_\theta - v) + w\cot\theta\,(u + v_\theta) - w_\theta(u + v\cot\theta)]$$

$$- \frac{4}{r^2}[wu_r + wv_r\cot\theta - w_r(u + v\cot\theta)]_\theta$$

$$- \frac{4}{r^2}[wu_r + wv_r\cot\theta - w_r(u + v\cot\theta)]$$

$$- \frac{4\cot\theta}{r^3}[w(u_\theta - v) + w(u + v_\theta)\cot\theta - w_\theta(u + v\cot\theta)],$$

$$S_1 = \left[4u_r^2 + \left(v_r - \frac{v}{r} + \frac{u_\theta}{r}\right)^2 + \left(w_r - \frac{w}{r}\right)^2\right]_r$$

$$+ \frac{1}{r^2}\left[\left(w_r - \frac{w}{r}\right)(w_\theta - w\cot\theta) - 2(u + v\cot\theta)\right.$$

$$\left.\times\left(v_r - \frac{v}{r} + \frac{u_\theta}{r}\right)\right]_\theta + \frac{1}{r}\left[8u_r^2 + \left(v_r - \frac{v}{r} + \frac{u_\theta}{r}\right)^2\right.$$

$$+ \left(w_r - \frac{w}{r}\right)^2 - \frac{4}{r^2}(u + v_\theta)^2 - \frac{4}{r^2}(u + v\cot\theta)^2$$

$$- \frac{2}{r^2}(w_\theta - w\cot\theta)^2 + \cot\theta\left(w_r - \frac{w}{r}\right)(w_\theta - w\cot\theta)$$

$$\left.- 2\cot\theta\,(u + v\cot\theta)\left(v_r - \frac{v}{r} + \frac{u_\theta}{r}\right)\right],$$

$$S_2 = \frac{1}{r}\left[\left(w_r - \frac{w}{r}\right)(w_\theta - w\cot\theta) - 2\left(v_r - \frac{v}{r} + \frac{u_\theta}{r}\right)\right.$$

$$\left.\times(u + v\cot\theta)\right]_r + \frac{1}{r}\left[\left(v_r - \frac{v}{r} + \frac{u_\theta}{r}\right)^2 + \frac{4}{r^2}(u + v_\theta)^2\right.$$

$$\left.+ \frac{1}{r^2}(w_\theta - w\cot\theta)^2\right]_\theta + \frac{3}{r^2}\left[\left(w_r - \frac{w}{r}\right)(w_\theta - w\cot\theta)\right.$$

$$\left.- 2\left(v_r - \frac{v}{r} + \frac{u_\theta}{r}\right)(u + v\cot\theta)\right] + \frac{\cot\theta}{r}$$

$$\times\left[\left(v_r - \frac{v}{r} + \frac{u_\theta}{r}\right)^2 + \frac{4}{r^2}(u + v_\theta)^2 - \left(w_r - \frac{w}{r}\right)^2\right.$$

$$\left.- \frac{4}{r^2}(u + v\cot\theta)^2\right],$$

$$S_3 = \frac{1}{r}\left[\left(v_r - \frac{v}{r} + \frac{u_\theta}{r}\right)(w_\theta - w\cot\theta) - 2(u + v_\theta)\left(w_r - \frac{w}{r}\right)\right]_r$$

$$+ \frac{1}{r}\left[\left(v_r - \frac{v}{r} + \frac{u_\theta}{r}\right)\left(w_r - \frac{w}{r}\right) - \frac{2}{r}u_r(w_\theta - w\cot\theta)\right]_\theta$$

$$+ \frac{3}{r^2}\left[v_r - \frac{v}{r} + \frac{u_\theta}{r}\right](w_\theta - w\cot\theta) + \frac{2\cot\theta}{r}$$

$$\times\left(v_r - \frac{v}{r} + \frac{u_\theta}{r}\right)\left(w_r - \frac{w}{r}\right) - \frac{6}{r^2}(u + v_\theta)\left(w_r - \frac{w}{r}\right)$$

$$- \frac{4}{r^2}v_r(w_\theta - w\cot\theta)$$

hwere the suffixes r, θ denote the partial derivation with respect to them.

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