GENERALISED ANGELESCU POLYNOMIALS: SOME PROPERTIES

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IN this paper, a new set of polynomials, $\pi_n^a(x)$, is introduced and their properties are studied.

Definition

$\pi_n^a(x)$ is defined as

$$\frac{\pi_n^a(x)}{(1+a)_n} = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{(1+a)_m} A_m(x) \tag{1}$$

and

$$\pi_n^a(0) = (1+a)_n \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{(1+a)_m} a_m. \tag{1 a}$$

Here

$$A_m(x) \equiv (a_0, a_1, \ldots, a_m, x, 1)^m. \tag{2}$$

Besides generalising the Angelescu polynomials, $\pi_n(x)$, defined[1] and studied[2-5] by earlier authors, some hitherto unpublished results concerning $\pi_n(x)$ are also derived. Many of the known relationships involving $\pi_n(x)$ are deduced as particular cases. It may be mentioned here that whereas many results proved earlier make use of the existence (or convergence) of the function $\sum_{r=0}^{\infty} a_r x^r/r!$, $|x| < x_0$, no such assumption is used here.

Two special cases

(i) For $a = 0$,

$$\pi_n^a(x) \equiv \pi_n(x)$$

(ii) For $a_r = 0$, $r \geqslant 1$; $a_0 = 1$,

$$\pi_n^a(x) = n!\ L_n^a(x). \tag{3}$$

Expansions in terms of Laguerre Polynomials

$$\pi_n^{\alpha+\beta+1}(x) = \sum_{m=0}^{n} \binom{n}{m} m!\, L_m^{\alpha}(x)\, \pi^{\beta}_{n-m}(0) \tag{4}$$

$$\pi_n^{\alpha+\beta+1}(x+y) = \sum_{m=0}^{n} \binom{n}{m} m!\, L_m^{\alpha}(x)\, \pi^{\beta}_{n-m}(y) \tag{5}$$

[(5) can be taken as the addition theorem as well.]

Some finite sums involving $\pi_n^{\alpha}(x)$

$$\pi_n^{\alpha}(x) = \sum_{m=0}^{n} \binom{n}{m} (\alpha-\beta)_m\, \pi^{\beta}_{n-m}(x) \tag{6}$$

$$\frac{\pi_n^{\alpha+1}(x)}{n!} = \sum_{m=0}^{n} \frac{\pi_m^{\alpha}(x)}{m!}. \tag{7}$$

A Recurrence Formula

$$\pi_n^{\alpha-1}(x) = \pi_n^{\alpha}(x) - n\pi_{n-1}^{\alpha}(x) \tag{8}$$

Results involving the derivatives

$$\frac{d}{dx}\left[\pi_n^{\alpha}(x) - \frac{\pi_{n+1}^{\alpha}(x)}{n+1}\right] = \pi_n^{\alpha}(x); \tag{9}$$

$$\frac{d}{dx}[\pi_n^{\alpha}(x)] = -\,n\pi_{n-1}^{\alpha+1}(x); \tag{10}$$

in particular,

$$\frac{d}{dx}\left[\pi_n(x) - \frac{\pi_{n+1}(x)}{n+1}\right] = \pi_n(x) \tag{11}$$

and

$$\frac{d}{dx}[\pi_n(x)] = -\,n\pi_{n-1}^{1}(x). \tag{12}$$

A Reciprocal Relationship

$$(1+\alpha)_n \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{(1+\alpha)_m} \pi_m^{\alpha}(x) = A_n(x) \tag{13}$$

is the result which can be seen to be ' the reciprocal ' of (1).

Some Identities

$$\sum_{r=0}^{n} \frac{\pi_r^{\alpha}(x)\, t^r f^r(z)}{r!\,(1+\alpha)_r} = \sum \frac{(-t)^r A_r(x) f^r(z+t)}{r!\,(1+\alpha)_r} \tag{14}$$

$$\sum_{m=0}^{n} \binom{n}{m} t^m \frac{\pi_m^{\alpha}(x)}{(1+\alpha)_m} \psi^*_{n-m}(z) = (1+t)^n \sum \binom{n}{m} \left(\frac{-t}{1+t}\right)^m \frac{A_m(x)}{(1+\alpha)_m} \psi^*_{n-m}\left(\frac{z}{1+t}\right) \tag{15}$$

where $\psi_n^*(z)$ is any polynomial of the form

$$\sum_{r=0}^{n} \binom{n}{r} \psi_r^0 z^r \text{ and } f(z), \text{ any polynominal in } z \text{ of degree } n. \tag{16}$$

In particular,

$$\sum \binom{n}{m} \frac{t^m (n-m)!}{(1+\alpha)_m (1+\beta)_{n-m}} \pi_m^{\alpha}(x) L^{\beta}_{n-m}(z) = (1+t)^n \sum \binom{n}{m} \left(\frac{-t}{1+t}\right)^m \frac{(n-m)!\, A_m(x)}{(1+\alpha)_m (1+\beta)_{n-m}} \times L^{\beta}_{n-m}\left(\frac{z}{1+t}\right), \tag{17}$$

and,

$$\sum \binom{n}{m} \frac{1}{(1+\alpha)_m} \left(\frac{\gamma \sin (\phi - \theta)}{\cos \phi \cos \theta}\right)^m \pi_m^\alpha (x)\, B_{n-m} (\gamma \tan \theta)$$

$$= \sum \binom{n}{m} \frac{1}{(1+\alpha)_m} \left(\frac{\gamma \sin (\theta - \phi)}{\cos \phi \cos \theta}\right)^m A_m (x)\, B_{n-m} (\gamma \tan \phi). \tag{18}$$

($B_n (x)$ = Bernoulli's polynomial)

$$\sum_{m=0}^{2l} \frac{(\tan \theta)^m}{m!\,(1+\alpha)_m} \pi_m^\alpha (x)\, P_{2l}^m (\sin \theta)$$

$$= (-1)^l \frac{(\tfrac{1}{2})_l}{l!} \sum_{m=0}^{l} \frac{(-l)_m\, (l+\tfrac{1}{2})_m}{(1+\alpha)_{2m}\, (2m)!} (2 \sin \theta)^{2m} A_{2m} (x) \tag{19}$$

$$\sum_{m=0}^{2l+1} \frac{(\tan \theta)^m}{m!\,(1+\alpha)_m} \pi_m^\alpha (x)\, P^m{}_{2l+1} (\sin \theta)$$

$$= (-1)^{l+1} \frac{(\tfrac{3}{2})_l}{l!} \sin \theta$$

$$\times \sum_{m=0}^{l} \frac{(-l)_m\, (l+\tfrac{3}{2})_m\, (2 \sin \theta)^{2m}}{(1+\alpha)_{2m+1}\, (2m+1)!} A_{2m+1} (x) \tag{20}$$

$$\sum_{m=0}^{n} \left(-\sqrt{\frac{1-z}{1+z}}\right)^m \frac{\pi_m^\alpha (x)\, P_n^m (z)}{m!\,(1+\alpha)_m}$$

$$= \sum_{m=0}^{n} \left(\frac{1-z}{2}\right)^m \frac{(-n)_m\, (n+1)_m}{m!\, m!\, (1+\alpha)_m} A_m (x) \tag{21}$$

$$\sum_{m=0}^{n} \left(-\sqrt{\frac{1-z}{1+z}}\right)^m \frac{\pi_m^n (x)\, P_n^m (z)}{m!\,(1+n)_m}$$

$$= \sum_{m=0}^{n} \left(\frac{z-1}{2}\right)^m \binom{n}{m} \frac{A_m (x)}{m!} \tag{22}$$

Some Integral Transforms

$$\int_0^\infty e^{-y} \pi_n^a(y)\, dy = e^{-x}\left[\pi_n^a(x) - n\pi^a_{n-1}(x)\right] \tag{23}$$

$$\int_0^\infty x^a e^{-x} \mathrm{L}_n(x)\, \pi_\mathrm{N}^{a+\beta+1}(x+y)\, dx$$

$$= \binom{\mathrm{N}}{n} \Gamma(a+n+1)\, \pi^\beta{}_{\mathrm{N}-n}(y), \quad \text{if } \mathrm{N} \geqslant n$$

$$= 0, \qquad \text{if } \mathrm{N} < n. \tag{24}$$

$$\int_0^\infty x^a e^{-x} \mathrm{L}_n^a(x)\, \pi_\mathrm{N}^{a+\beta+1}(x)\, dx$$

$$= \binom{\mathrm{N}}{n} \Gamma(a+n+1)\, \pi^\beta{}_{\mathrm{N}-n}(0), \quad \text{if } \mathrm{N} \geqslant n$$

$$= 0, \qquad \text{if } \mathrm{N} < n. \tag{24 a}$$

$$\int_0^\infty x^{\gamma-1} e^{-x} \pi_\mathrm{N}^{a+\beta+1}(x+y)\, dx = \Gamma(\gamma)\, \pi_\mathrm{N}^{1+a+\beta-\gamma}(y) \tag{25}$$

$$\int_0^\infty x^{\gamma-1} e^{-x} \pi_\mathrm{N}^{a+\beta+1}(x)\, dx = \Gamma(\gamma)\, \pi_\mathrm{N}^{1+a+\beta-\gamma}(0) \tag{25 a}$$

§ 2. We present here the proof of the identity given by (14), as many of the results cited above can be deduced from this. Others can be proved by making use of (1) and (2).

To prove (14), we expand $f^{(r)}(z+t)$ as $\sum_{s=0}^{n-r} f^{r+s}(z)\, t^s/s!$. Rearranging, the coefficient of t^m in the right-hand side of (14) is

$$\sum \frac{(-1)^r \mathrm{A}_r(x)}{r!\,(1+a)_r} \frac{f^{(m)}(z)}{(m-r)!}$$

$$= \frac{f^m(z)}{m!} \sum \binom{m}{r} \frac{(-1)^r}{(1+a)_r} \mathrm{A}_r(x)$$

$$= \frac{f^m(z)}{m!\,(1+a)_m} \pi_m^a(x) \quad \text{from (1)}$$

$$= \text{left-hand side of (14).}$$

Hence the identity.

It can be easily shown that many of the results derived in earlier works[2-5] follow immediately as particular cases from the equations stated above. As illustrations, we cite a few.

$$\mathrm{D}^p [\pi_{n-r}(x)] = \frac{d^p}{dx^p} [\pi_{n-r}(x)]$$

$$= (-1)^p \frac{(n-r)!}{(n-r-p)!} \pi^p{}_{n-r-p}(x) \quad \text{from (10)}$$

and hence,

$$\sum_0^n \binom{n}{r} r!\ \mathrm{L}_r(x)\, \mathrm{D}^p [\pi_{n-r}(x)]$$

$$= (-1)^p \frac{n!}{(n-p)!} \sum_0^n \binom{(n-p)}{r} r!\ \mathrm{L}_r(x)\, \pi^p{}_{n-p-r}(x)$$

$$= (-1)^p \frac{n!}{(n-p)!} \pi_{n-p}{}^{p+1}(2x) \quad \text{from (5).} \tag{26}$$

Similarly,

$$\sum_0^n \binom{n}{r} (n-r)!\ \pi_r(x)\, \mathrm{D}^p [\mathrm{L}_{n-r}(x)]$$

$$= (-1)^p \sum_0^n \binom{n}{r} (n-r)!\, \pi_r(x)\, \mathrm{L}^p{}_{n-p-r}(x)$$

$$= \frac{(-1)^p\, n!}{(n-p)!} \sum_0^n \binom{n-p}{r} (n-p-r)!\ \pi_r(x)\, \mathrm{L}^p{}_{n-p-r}(x)$$

$$= \frac{(-1)^p n!}{(n-p)!} \pi_{n-p}{}^{p+1}(2x), \text{ again, from (5).} \tag{27}$$

Hence, it is proved that the following reciprocal relationships, *viz.*,

$$\sum_{0}^{n} \binom{n}{r} r!\ L_r(x)\ D^p\ [\pi_{n-r}(x)]$$

$$= \sum_{0}^{n} \binom{n}{r} \pi_r(x)\ D^p\ [(n-r)!\ L_{n-r}(x)] \tag{28}$$

is true.

Equation (28) can be easily generalised as

$$\sum \binom{n}{r} r!\ L_r^{\alpha}(x)\ D^p\ \pi^{\beta}_{n-r}(y)$$

$$= (-1)^p \frac{n!}{(n-p)!} \pi_{n-p}^{\alpha+\beta+p+1}(x+y)$$

$$= \sum \binom{n}{r} \pi_r^{\alpha}(x)\ D^p\ [(n-r)!\ L^{\beta}_{n-r}(y)];\ \left(D = \frac{d}{dy}\right) \tag{29}$$

(28) has been proved* by Varma[3] and Shastri[4] by assuming the existence of the generating function $\Sigma\, \pi_r(x)\, t^r/r!$.

(It may be mentioned that these relations hold good even if the infinite series $\Sigma\, \pi_r(x)\, t^r/r!$ is divergent.)

$$\sum_{0}^{n} \frac{\pi_r(u)}{r!} L_{n-r}(v) - \sum_{0}^{n-1} \frac{\pi_r(u)}{r!} L_{n-r-1}(v)$$

$$= \frac{\pi_n{}^1(u+v)}{n!} - \frac{\pi_{n-1}{}^1(u+v)}{(n-1)!}$$

$$= \frac{\pi_n(u+v)}{n!} \text{ from (5) and (8)} \tag{30}$$

and this is the addition theorem suggested by Varma,[3] who has assumed the convergence of $\Sigma\, \pi_r(x)\, t^r/r!$.

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* There is a slight change in the notation used by these authors and that used in this paper. $r!\, L_r^{\alpha}(x)$ of this paper corresponds to $L_r(x)$ of Reference (3).

A generalisation of (30), given by Shastri,[4] can also be proved to follow easily from the equations (5) and (6).

$$\sum_{r=0}^{m} (-1)^r \binom{-n-1}{r} \frac{\pi_{m-r}(x+y)}{(m-r)!}$$

$$= \frac{1}{n!} \sum_{r=0}^{m} \binom{n}{r} (n+1)_r \, \pi_{m-r}(x+y)$$

$$= \frac{\pi_m^{n+1}(x+y)}{n!}, \text{ by (6)}$$

$$= \sum_{r=0}^{m} L_{m-r}(x) \frac{\pi_r(y)}{r!}, \text{ by (5) (\textit{cf.} Ref. 4).} \quad (31)$$

Since

$$\underset{a \to 1-n}{\mathrm{Lt}} \frac{\Phi(a, c; x)}{\Gamma(c)} = \frac{(a)_n}{n!} x^n \Phi(a+n, 1+n; x) \text{ (Ref. 6)}$$

$$L_n^{-1}(x) = \left(-\frac{x}{n}\right) L^1_{n-1}(x) \quad (32)$$

where $L_n^{\alpha}(x)$ is defined as $\binom{n+\alpha}{n} \Phi(-n, 1+\alpha; x)$.

In (5), assuming $\alpha = -1$ and using (4), we obtain,

$$\pi_n^{\beta}(0) - \pi_n^{\beta}(x) = \sum_{r=0}^{n-1} \frac{n!}{r!} \pi_r^{\beta}(0) \frac{x\, L^1_{n-r-1}(x)}{(n-r)} \quad (33)$$

—this generalises the one proved by Shastri (for $\beta = 0$) by the method of operational calculus, so that we conclude that,

$$\int_0^{\infty} e^{-x} L_m^1(x) [\pi_n^{\beta}(0) - \pi_n^{\beta}(x)]\, dx$$

$$= 0, \quad \text{if } m \geqslant n,$$

$$= \frac{n!}{(n-m-1)!} \pi^{\beta}_{n-m-1}(0), \text{ if } 0 \leqslant m \leqslant n-1. \quad (34)$$

For $\beta = 0$, this reduces to the known result.[4]

Many such similar integrals and finite sums involving the generalised Angelscu polynomials can be obtained but are not presented here for the sake of brevity.

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