SMALL AMPLITUDE HYDROMAGNETIC WAVES IN COMPRESSIBLE MEDIUM WITH HALL CURRENT

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1. INTRODUCTION

THE propagation of small amplitude waves in a conducting medium has been initiated by Herlofson[3] and studied in detail by Van de Hulst[5] and Bhatnagar.[1] However, certain modifications are necessary to apply this theory to propagation in a fully-ionized high temperature gas embedded in a uniform magnetic field. This has been pointed out by Cowling,[2] who stressed the importance of electron partial pressure and the magnetic induction due to transverse Hall current. Recently, Ong[4] has considered this problem and found that there are three possible waves for the propagation along the primitive magnetic field.

We have studied the propagation of the waves in an arbitrary direction in relation to uniform external field. At any inclination, except the degenerate case of transverse to the magnetic field, there are three elliptically polarized waves, the first one propagating mainly the magnetic energy, the second (a modified sound wave) propagating thermal energy and the last one which can be propagated only when the frequency of the wave is below a critical frequency depending upon the inclination of propagation. Hall current introduces a medium-like anisotropy into the gas and is responsible for this behaviour of the third wave. We calculate the amplitudes and relative kinetic and magnetic energies, and find that as this critical frequency is approached, the elements of gas attain enormous velocities thereby completely destroying the wave motion. Also, it is found that the kinetic as well as the magnetic energy is equally divided into transverse and longitudinal part in this case. We have also studied this graphically for the physical conditions prevailing in H-II clouds and Galactic Halo.

2. Basic Equations

Following Ong, we shall assume that the temperature distribution is uniform and wave propagation takes place isothermally so that the effect of electron pressure is nil. The linearized equations governing the motion are:

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} \vec{v} = 0,$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = - S^2 \operatorname{grad} \rho + \frac{1}{4\pi} (\operatorname{curl} \vec{b}) \times \vec{B}_0,$$

and

$$\frac{\partial \vec{b}}{\partial t} = \operatorname{curl} (\vec{v} \times \vec{B}_0) - \frac{1}{4\pi n e} \operatorname{curl} (\operatorname{curl} \vec{b} \times \vec{B}_0), \tag{2.1}$$

where ρ_0, $\vec{B}_0$ are the undisturbed density and magnetic field, $\rho_0 + \rho$, $\vec{v}$, $\vec{B}_0 + \vec{b}$ are the perturbed density, velocity and magnetic field, while S is the sound speed and n the uniform electron density. We may note that the second term on the right-hand side of the third equation in (2.1) arises from Hall current.

To study the plane wave propagation, we seek for a solution whose space and time dependence are given by $\exp\,[i\,(\vec{k}.\vec{r} + \omega t)]$. Also, we introduce the following non-dimensional parameters to facilitate our analysis:

$$\rho' = \frac{\rho}{\rho_0}, \quad \vec{v}' = \frac{\vec{v}}{A}, \quad \vec{b}' = \frac{\vec{b}}{B_0},$$

$$W = \frac{\omega}{\omega_i}, \quad \vec{K} = \frac{A\vec{k}}{\omega_i} \tag{2.2}$$

where

$$A = \sqrt{\frac{{B_0}^2}{4\pi\rho_0}}, \text{ the Alfven's velocity}$$

and

$$\omega_i = \frac{B_0 e}{M}, \text{ the ion Larmor frequency.}$$

Instead of proceeding directly to obtain the dispersion relation, we shall first solve for the amplitudes so as to enable us to understand the behaviour of the waves that are possible. Without loss of generality, we choose the z-axis along the primitive magnetic field and the xoz plane as the plane of primitive magnetic field and direction of propagation. Then we have

$$v_x = a^2 \mathrm{WKD} \sin\theta\,(\mathrm{W}^2 - \mathrm{K}^2\cos^2\theta - \mathrm{K}^4\cos^2\theta)\,\rho'$$

$$v_y = -\,ia^2\mathrm{K}^5\mathrm{D}\sin\theta\cos^2\theta\,\rho'$$

$$v_z = -\,\frac{a^2\mathrm{K}}{\mathrm{W}}\cos\theta\rho'$$

$$b_x = a^2\mathrm{K}^2\mathrm{D}\sin\theta\cos\theta\,(\mathrm{W}^2 - \mathrm{K}^2\cos^2\theta)\,\rho'$$

$$b_y = -\,ia^2\mathrm{WK}^4\mathrm{D}\sin\theta\cos\theta\rho'$$

and

$$b_z = -\,a^2\mathrm{K}^2\mathrm{D}\sin^2\theta\,(\mathrm{W}^2 - \mathrm{K}^2\cos^2\theta)\,\rho', \tag{2.3}$$

where

$$\mathrm{D} = [\mathrm{K}^4\cos^2\theta\,(\mathrm{W}^2 - 1) + \mathrm{K}^2\mathrm{W}^2(1 + \cos^2\theta) - \mathrm{W}^4]^{-1}.$$

Substituting these in continuity equation, we obtain the dispersion relation (same as that obtained by Ong):

$$x^3 - x^2(1 + a^2 + \cos^2\theta) + x\cos^2\theta\,(1 + 2a^2 - \mathrm{W}^2) + a^2\cos^2\theta\,(\mathrm{W}^2 - \cos^2\theta) = 0 \tag{2.4}$$

where

$$a^2 = \frac{\mathrm{S}^2}{\mathrm{A}^2}$$

and

$$x = \frac{\mathrm{W}^2}{\mathrm{K}^2},$$ the square of the wave velocity.

3. DISCUSSION

The discriminant of the cubic equation (2.4) is

$$\triangle = -\,\frac{\cos^2\theta}{27}\,[4\mathrm{W}^2\{\mathrm{W}^2\cos^2\theta + \sin^2\theta - (1 - a^2)^2\}^2 + 4\mathrm{W}^2\{(1 - a^2)^2 + 6\,(a^2 - \cos^2\theta)^2 + (a^2 + 3\cos^2\theta)\sin^2\theta\}\,a^2\sin^2\theta + \sin^4\theta\cos^2\theta\,(\mathrm{W}^2 + a^2 - 1)^2 + 4a^2\sin^2\theta\cos^2\theta\,(\mathrm{W}^2 - \sin^2\theta)^2]. \tag{3.1}$$

Thus, it is negative definite for all angles $0 \leq \theta < \pi/2$. Hence, the cubic equation (2.4) has three real roots two of which are always positive. The third root is also positive if the applied frequency is less than the critical frequency ω_c given by

$$\omega_c = \omega_i \cos\theta; \tag{3.2}$$

otherwise it is negative. The three roots are approximately given by

$$x_0 = \frac{1 + a^2 + \cos^2\theta}{3} - \frac{G}{H}, \tag{3.3}$$

$$x_{\pm} = \frac{1 + a^2 + \cos^2\theta}{3} + \frac{G}{6H} \pm \sqrt{3}\,(-H)^{\frac{1}{2}}, \tag{3.4}$$

and

$$x_- = \frac{1 + a^2 + \cos^2\theta}{3} + \frac{G}{6H} - \sqrt{3}\,(-H)^{\frac{1}{2}}, \tag{3.5}$$

where

$$G = \frac{1}{27}\,[27a^2\cos^2\theta\,(W^2 - \cos^2\theta) + 9\cos^2\theta\,(1 + 2a^2 - W^2) \times (1 + a^2 + \cos^2\theta) - 2\,(1 + a^2 + \cos^2\theta)^3]$$

and

$$-H = \frac{1}{9}\,[3\cos^2\theta\,W^2 + (a^2 - \cos^2\theta)^2 + (2a^2 + 1)\sin^2\theta]. \tag{3.6}$$

Concentrating on the behaviour of the waves for large and small frequencies, we find that

$$x_0 \to a^2, \quad x_{\pm} \to \pm W\cos\theta, \quad \text{as } \omega \to \infty, \tag{3.7}$$

and

$$x_0 \to \cos^2\theta,$$
$$x_{\pm} \to \tfrac{1}{2}\,[(1 + a^2) \pm \{(1 + a^2)^2 - 4a^2\cos^2\theta\}^{\frac{1}{2}}], \tag{3.8}$$

as

$$\omega \to 0.$$

Also we note here that if $\omega > \omega_c$

$$x_- < 0. \tag{3.9}$$

For the degenerate case of propagation transverse to the magnetic field, we have

$$x_0 = x_- = 0$$
$$x_+ = (1 + \alpha^2). \tag{3.10}$$

Thus we have three waves, x_+, modified Alfven's wave (MAW), x_0, modified Acoustic wave (MSW) and x_-, modified decaying Alfven's wave (MDAW).

MAW.—This is a highly coupled magnetohydrodynamic and acoustic wave, the wave velocity of which depends essentially on the direction of propagation. For low frequencies, it has the value A for $\theta = 0$ and slowly increases as the inclination increases and reaches the value $\sqrt{A^2 + S^2}$ transverse to the magnetic field. For the high frequencies the velocity increases as $\sqrt{W \cos \theta}$.

Further, we have the kinetic energy

$$\begin{aligned} V &= v_x v_x^* + v_y v_y^* + v_z v_z^* \\ &= \frac{\alpha^4 \rho'^2 D^2}{x^2} [\sin^2\theta \, (W^2 \cos^2\theta + x \cos^2\theta - x^2)^2 \\ &\quad + \cos^2\theta \, \{\cos^2\theta \, (W^2 - 1) + x\,(1 + \cos^2\theta) - x^2\}^2 \\ &\quad + \sin^2\theta \cos^4\theta W^2] \end{aligned} \tag{3.11}$$

and the magnetic energy

$$\begin{aligned} M &= b_x b_x^* + b_y b_y^* + b_z b_z^* \\ &= \alpha^4 \rho'^2 D^2 [\sin^2\theta \, (\cos^2\theta - x)^2 + \cos^2\theta \sin^2\theta W^2], \end{aligned} \tag{3.12}$$

where * denotes the complex conjugate of the quantity.

Thus, it is quite evident that as $\omega \to \infty$, for the MAW,

$$V \to 0$$

and

$$M \to \frac{2\alpha^4 \rho'^2 \sin^2\theta}{(1 + \cos^2\theta)^2} \cdot \tag{3.13}$$

Hence, for large frequencies there is definite energy propagation in the form of electromagnetic radiation, which is evidently maximum when the propagation is transverse to the primitive magnetic field and tends to zero when wave propagates along the magnetic field. So we conclude that electromagnetic radiation is the main vehicle for the energy to escape from the gas.

MSW.—This wave, for small frequencies, starts with sound speed along the primitive magnetic field. In any other direction it starts as a retarded Alfven wave with speed A $\cos\theta$ and decreases slowly as the inclination increases and becomes zero transverse to the magnetic field. Its velocity is greater than the sound speed if

$$\cos^2\theta > \alpha^2$$

and less than the sound speed if

$$\cos^2\theta < \alpha^2.$$

It reduces to a pure sound wave if $\alpha^2 = \cos^2\theta$. Consequently, it presents an interesting feature when $\alpha < 1$, namely, it changes from fast to slow as the inclination crosses $\theta_c = \cos^{-1}\alpha$. Another interesting feature is that, whatever be the velocity it starts with, it tends to sound speed as the frequency becomes large. Hence compressibility of the medium is mainly responsible for its propagation. It is natural to expect that near the primitive magnetic field its velocity is inhibited and farther away increased. Also, we find from energy considerations for longitudinal oscillations using the expressions (3.11) and (3.12), that

$$M \rightarrow 0$$

and

$$V \rightarrow \alpha^2 (\rho')^2 \cos^2\theta \tag{3.15}$$

as $\omega \rightarrow \infty$.

Hence for higher frequencies, it is the main carrier of kinetic energy and the propagation of kinetic energy is maximum along the magnetic field as the wave is purely longitudinal as seen from (2.3) and is zero in propagation transverse to the magnetic field.

MDAW.—This is by far the most interesting since it reveals the medium-like behaviour of the gas due to the inclusion of the effect of Hall current. Already we have pointed out that this is completely damped out for the frequencies higher than the critical frequency (3.2). The energy considerations yield a qualitative explanation of this. From the expressions (3.11) and (3.12) for V and M it is evident that as the critical frequency $\omega_i \cos\theta$ is approached, wave velocity tends to zero and the kinetic energy becomes extremely large and each one of the velocity components also becomes large, while the magnetic energy remains finite. Thus as the frequency approaches the critical frequency, the elements of the gas assume large velocities and this completely disrupts the wave motion and hence the damping. However, one remarkable feature of this wave is that

$$\frac{v_x v_x^* + v_z v_z^*}{v_y v_y^*} \rightarrow 1$$

and

$$\frac{b_x b_x^* + b_z b_z^*}{b_y b_y^*} \rightarrow 1 \tag{3.16}$$

as the critical frequency is approached, so that the total energy is equally distributed between transverse and longitudinal components. It may be noted that the above conclusions are perhaps the outcome of linearization and one should verify them on the basis of non-linear theory.

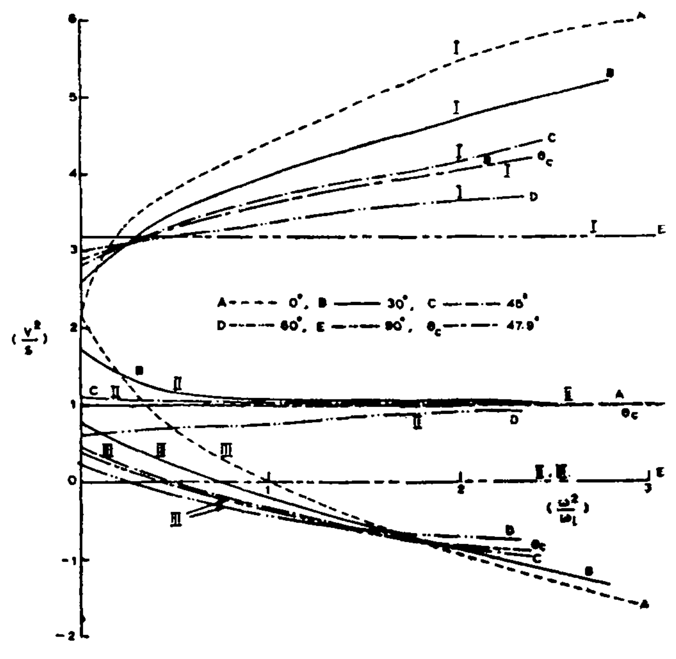

FIG. 1. H-II cloud $a^2 = 0{\cdot}45$, plot of square of wave velocity against square of frequency.

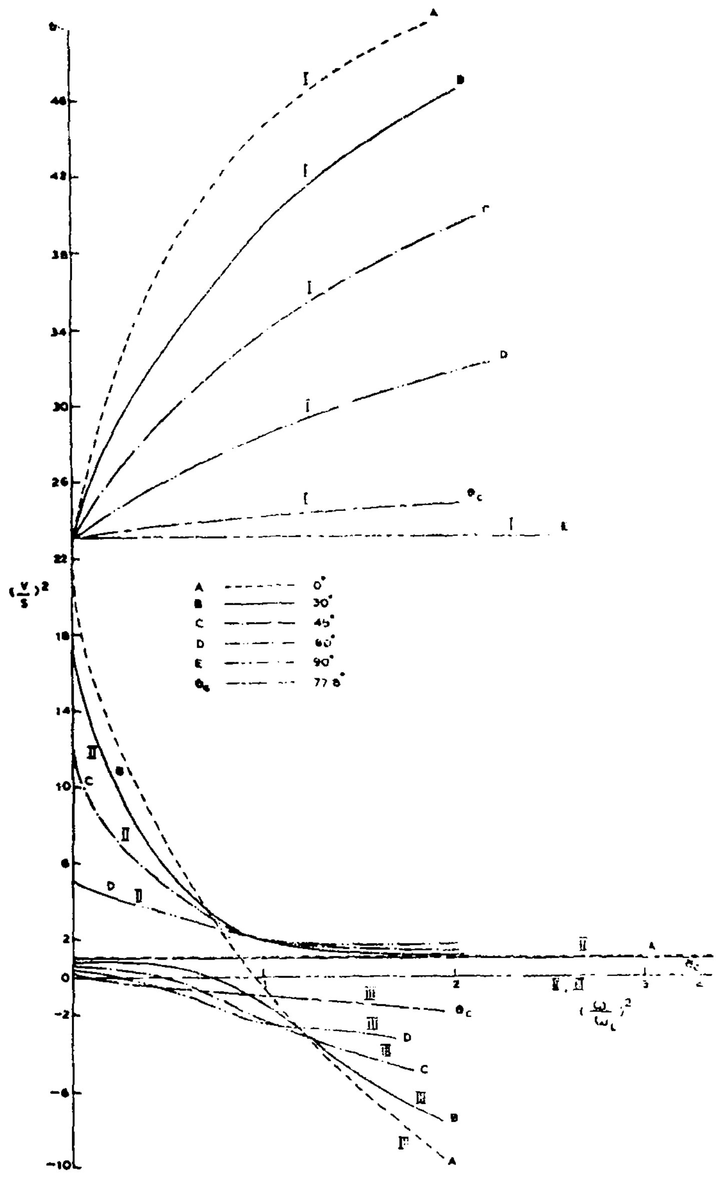

FIG. 2. Galactic Halo, $\alpha^2 = 0{\cdot}045$, plot of square of wave velocity against square of frequency.

4. We have studied the behaviour of the three waves for the physical conditions prevailing in H-II clouds and Galactic Halo:

	H-II cloud	Galactic Halo
Magnetic field B_0 (in Gauss)	10^{-5}	10^{-5}
Particle density n (no./cm.^{3})	1	10^{-3}
Density (gm./cm.^{3})	$1 \cdot 7 \times 10^{-24}$	$1 \cdot 7 \times 10^{-27}$
Square of Alfven speed A^2 (cm.^{2}/sec.^{2}) ..	5×10^{12}	5×10^{15}
Sound speed S (cm./sec.)	$1 \cdot 5 \times 10^{6}$	$1 \cdot 5 \times 10^{7}$
α^2	$0 \cdot 45$	$0 \cdot 045$

In Figs. 1 and 2 we have plotted (x/α) against W for H-II cloud and Galactic Halo respectively. The shape of the curves lends justification to the theoretical discussion of Section 3.

We also record the critical frequency beyond which the third wave becomes highly damped, the damping time being inversely proportional to 3/2 power of the frequency, for various values of θ.

θ	0°	30°	45°	47°, 9′	60°	75°	77°, 8′	80°
ω_e	0·0943	0·0817	0·0667	0·0633	0·0471	0·0244	0·02	0·0164.

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