A GENERALIZED SQUARE YIELD CONDITION
FOR SHELLS OF REVOLUTION

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INTRODUCTION

The plastic analysis of a general rotationally symmetric shell is extremely difficult and relatively little work has been done. The yield surface for a shell whose material obeys the Tresca yield condition was derived by Onat and Prager. The corresponding equations for a shell material obeying the Mises' yield condition were obtained by Hodge. These two yield surfaces are non-linear and the mathematical difficulties are formidable while determining the carrying capacities of structures. Two linear approximations have been proposed by Hodge. In Reference the yield condition for an ideal sandwich shell is used as an approximation for the uniform shell and Reference uses a linear surface which circumscribes the yield surface based on the Tresca condition. Theoretically, it is possible to obtain an exact solution to a rotationally symmetric shell problem by using the above piece-wise linear yield conditions. In practice, the equations become quite complex, and it is doubtful if the resulting labour is worthwhile. In this paper, a much simpler yield surface is proposed. Based on this yield surface, the concentrated collapse load of a simply supported spherical cap is determined.

BASIC EQUATIONS

Figure 1 shows a rotationally symmetric shell element whose state of stress is described by four generalized stresses: The circumferential and meridional bending moments $M_\theta$ and $M_\phi$ and the circumferential and meridional membrane forces $N_\theta$ and $N_\phi$. The shear force $S$ is not considered to be a generalized stress but has the nature of a reaction. The load per unit area of the middle surface of the shell has the components $P_\theta$ in the direction of the meridian and $P_\rho$ in the normal direction. The element has the distance $R_\rho$ from the axis of revolution and its principal radii of curvature are $R_1$ and $R_2$. The generalized stresses must satisfy three equations of equilibrium which may be written
Fig. 1. Element of shell of revolution.

\[ (r_0 n_\phi)' - r_1 n_\theta \cos \phi - r_0 s + r_0 r_1 P_\phi = 0 \]
\[ r_0 n_\phi + r_1 n_\theta \sin \phi + (r_0 s)' + r_0 r_1 P_\tau = 0 \]
\[ k [(r_0 m_\phi)' - r_1 m_\theta \cos \phi] - r_0 r_1 s = 0 \]  

(1)

where we have defined \( n = N/N_0 = N/2\sigma_0 H, \) \( m = M/M_0 = M/\sigma_0 H^3, \)
\( s = S/N_0, \) \( p_\phi = L P_\phi/N_0, \) \( p_\tau = L P_\tau/N_0, \) \( r = R/L; \) \( \sigma_0 \) is the tensile yield stress of the material. The shell is of uniform thickness \( 2H \) and radius \( R \) and \( L \) is a typical length. Primes denote differentiation with respect to \( \phi. \)

The state of strain is described by four generalized strains which may, in turn, be expressed in terms of the meridional and normal components of the displacement \( V \) and \( W. \) The generalized strain rates and the velocities are related by

\[ \epsilon_\theta = \frac{1}{r_2} (v \cot \phi - \dot{w}), \quad \epsilon_\phi = \frac{1}{r_1} (\dot{v} - \dot{w}) \]
\[ x_\theta = -\frac{k \cot \phi}{r_1 r_2} (v + \dot{w}'), \quad x_\phi = -\frac{k}{r_1} (\dot{v} + \dot{w}')'. \]  

(2)

**YIELD CONDITION AND FLOW RULE**

In order to formulate the shell problem, all the constituent equations have to be expressed in terms of the generalized stresses and strain rates.
The Equations of Equilibrium (1) and the relations (2) between strain rates and velocities have been derived by direct consideration of an infinitesimal element of the shell. In the case of an elastic material, the relations between generalized stresses and strain rates are obtained by using Hooke's Law.

For a perfectly plastic material, however, it is necessary to express the yield condition in terms of generalized stresses. Onat and Prager\(^1\) have obtained the yield surface for shells where the material satisfies Tresca's yield condition of maximum shear. Hodge\(^2\) has derived the yield surface for a shell whose material satisfies Mises' yield condition of maximum octahedral stress. Some special cases of these yield surfaces are shown in Figs. 2, 3 and 4. Hodge\(^3,4\) has also proposed two piece-wise linear approximations for the yield surface for a rotationally symmetric shell. However, relatively little work has been done on the general problem of rotationally symmetric shells due to the inherent complexity of the equations. Moreover, the available experimental evidence indicates that it is doubtful whether the most painstaking theoretical analysis could predict the actual behaviour of structures in the plastic range and it seems justifiable to choose a much simpler approximation than the ones that have been suggested so far. Hence we consider a linear surface in four-dimensional space defined by the twelve hyperplanes listed in Table I. The appropriate special cases are shown in Figs. 2-4.

**Table I**

*Generalized square yield condition and the associated flow rule*

<table>
<thead>
<tr>
<th>Face</th>
<th>Equation</th>
<th>Strain-rate vector ((\xi_\theta, \xi_\phi, x_\theta, x_\phi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(n_\theta = 1)</td>
<td>(\mu (1, 0, 0, 0))</td>
</tr>
<tr>
<td>2</td>
<td>(n_\phi = 1)</td>
<td>(\mu (0, 1, 0, 0))</td>
</tr>
<tr>
<td>3</td>
<td>(-n_\theta = 1)</td>
<td>(\mu (-1, 0, 0, 0))</td>
</tr>
<tr>
<td>4</td>
<td>(-n_\phi = 1)</td>
<td>(\mu (0, -1, 0, 0))</td>
</tr>
<tr>
<td>5</td>
<td>(m_\theta = 1)</td>
<td>(\mu (0, 0, 1, 0))</td>
</tr>
<tr>
<td>6</td>
<td>(m_\phi = 1)</td>
<td>(\mu (0, 0, 0, 1))</td>
</tr>
<tr>
<td>7</td>
<td>(-m_\theta = 1)</td>
<td>(\mu (0, 0, -1, 0))</td>
</tr>
<tr>
<td>8</td>
<td>(-m_\phi = 1)</td>
<td>(\mu (0, 0, 0, -1))</td>
</tr>
</tbody>
</table>
From Figs. 2–4, it is seen that the proposed yield surface is a circumscribed linear approximation to the Tresca yield condition. Hence the collapse load based on the above yield surface is an upper bound to that obtained on the basis of the Tresca yield condition. From Fig. 2 it is also seen that for the case of a circular cylindrical shell without end load the proposed yield surface coincides with the square yield condition first suggested by Drucker. Further, the projections of the four-dimensional yield surface, Table I on the \((m_\theta, m_\phi), (n_\theta, n_\phi), (m_\theta, n_\phi)\) and \((m_\phi, n_\theta)\) planes are all seen to be squares. Hence it seems appropriate to call the yield surface listed in Table I as a “GENERALIZED SQUARE YIELD CONDITION”. Two two-dimensional projections of the above yield condition are shown in Fig. 5.
The plastic potential flow rule states that if the stress point is in the interior of the yield surface, then the strain rate vector
\[ E(\epsilon_{\theta}, \epsilon_{\phi}, x_{\theta}, x_{\phi}) = 0; \]
if the stress point is in contact with one of the hyperplanes of the yield surface, then \( E \) is directed along the outward normal of that hyperplane; if the stress point is at the intersection of two or more hyperplanes, then \( E \) must be a linear combination with non-negative coefficients of the outward normals to the hyperplanes involved; for the considered rigid perfectly plastic material, the stress point can never be outside the yield surface. The simplicity achieved by using the proposed yield condition will now be illustrated by an example.
Fig. 4. Yield Surfaces for arches.

- Tresca, Mises conditions.
- Generalized square condition.

Fig. 5. Generalized square yield condition.
Simply Supported Spherical Cap Under a Concentrated Load at the Vertex

If the typical length of the shell is taken to be its radius, the dimensionless radii of the sphere are

\[ r_1 = r_2 = 1, \quad r_o = \sin \phi. \quad (3) \]

It is not possible to make a direct formal analysis of a shell subjected to a concentrated load since the resulting shear force at the point of load application is infinite. Hence we consider a shell where the load is applied over a small but finite area and to get the "Solution for Concentrated Load" we pass to the limit as the loaded area tends to zero.

Figure 6 shows a simply supported spherical cap loaded over a portion of the surface. In view of Eqs. (3), the Equations of Equilibrium (1) for the loaded portion of the shell become

\[
\begin{align*}
(n_\phi \sin \phi)' - n_\theta \cos \phi &= s \sin \phi \\
(s \sin \phi)' + \sin \phi (p + n_\theta + n_\phi) &= 0 \\
k [(m_\phi \sin \phi)' - m_\theta \cos \phi] &= s \sin \phi.
\end{align*}
\]

\text{(4 a)}
For the portion of the shell on which there is no pressure acting, the corresponding equations are

\[
(n_{\phi} \sin \phi)' - n_{\theta} \cos \phi = s \sin \phi \\
(s \sin \phi)' + (\sin \phi) \times (n_{\theta} + n_{\phi}) = 0 \\
k [(m_{\phi} \sin \phi)' - m_{\theta} \cos \phi] = s \sin \phi.
\]

The relations (2) between the velocities and strain rates become

\[
\varepsilon_{\theta} = v \cot \phi - w, \quad \varepsilon_{\phi} = v' - w \\
x_{\theta} = -k (\cot \phi) [(v + w')'], \quad x_{\phi} = -k (v + w')'.
\]

At the centre of the shell, isotropy and symmetry demand that

\[
\phi = 0 : n_{\theta} = n_{\phi}, \quad m_{\theta} = m_{\phi}, \quad s = 0
\]

and either \(w' = v = 0\) or there is a hinge circle.

At the edge of the simply supported cap

\[
\phi = \alpha : m_{\phi} = 0, \quad w = 0
\]

and either \(v = 0\) or there is a hinge circle.

A hinge circle is a circle across which \(w'\) and/or \(v\) are discontinuous. A discontinuity in \(w'\) is possible if \(|m_{\phi}| = 1\) and a discontinuity in \(v\) is possible if \(|n_{\phi}| = 1\).

At \(\phi = \beta, m_{\phi}, n_{\phi}, s, v, w\) and \(w'\) must be continuous.

The general procedure for finding the collapse load of a structure is first to make a hypothesis for the stress profile. For any such hypothesis the equilibrium equations and flow rule are solved. The resulting stress and velocity fields must then be examined. The solution will be statically admissible if the stress profile lies everywhere on or inside the yield surface. It will be kinematically admissible if the strain-rate vector is directed outward to the yield surface and lies between the appropriate limits at a corner. Solutions which are statically or kinematically admissible will provide lower or upper bounds, respectively, while the actual solution is distinguished by the fact that it must be both statically and kinematically admissible.

When the load is applied over the entire surface of the cap the stress profile was found to correspond to the hyperplanes \(n_{\phi} = -1\) and \(m_{\theta} = 1\), Reference[4]. These hyperplanes correspond to regime 45 of the generalized square yield condition. Hence for values of \(\beta\) slightly less than \(\alpha\) it is
reasonable to expect the same stress profile to be valid. The substitution
of the above equations of the hyperplanes into the Equations of Equilibrium
(4) and the use of the boundary and continuity conditions (6) results in

\[ 0 \leq \phi \leq \beta : n_\phi = -1, \ m_\theta = 1 \]

\[ n_\theta = - \frac{1}{2} \left[ p - (p - 2) \sec^2 \phi \right] \]  
(7)

\[ m_\phi = 1 - \frac{(p - 2)}{2k} \left[ \frac{1}{\sin \phi} \log(\sec \phi + \tan \phi) - 1 \right] \]

\[ \beta \leq \phi \leq \alpha : n_\phi = -1 ; \ m_\theta = 1 \]

\[ n_\theta = - \frac{\cos^2 \beta}{2} \left[ p - (p - 2) \sec^2 \beta \right] \sec^2 \phi \]

\[ m_\phi = \left( 1 - \frac{1}{k} \right) \left( 1 - \frac{\sin a}{\sin \phi} \right) - \frac{(p \cos^2 \beta - (p - 2))}{2k \sin \phi} \]

\[ \times \log \frac{\sec a + \tan a}{\sec \phi + \tan \phi} . \]  
(8)

Using the condition that \( m_\phi \) must be continuous at \( \phi = \beta \), the collapse
pressure \( p \) is found to be

\[ p = \frac{2 \left[ \log(\sec a + \tan a) - \sin a \right] + k \sin a}{\log(\sec a + \tan a) - \sin \beta - \cos^2 \beta \log \frac{\sec a + \tan a}{\sec \beta + \tan \beta}} . \]  
(9)

The solution given by equations (7), (8) and (9) will be statically
admissible if the stress profile remains on the finite faces 4 and 5. This means
that the stresses must satisfy

\[ -1 \leq n_\theta \leq 1 ; \ -1 \leq m_\phi \leq 1 . \]  
(10)

The discussion of the above inequalities is facilitated by expanding the
bracketed term in the numerator of Eqn. (9) \( \log(\sec a + \tan a) - \sin a \)
\[ = \frac{1}{3} \sin^3 a + \frac{1}{6} \sin^5 a + . \]

As regards the range \( 0 \leq \phi \leq \beta \), it can be verified that the four inequalities
(10) are satisfied provided

\[ p \leq 2 + \frac{4k \sin \beta}{\log(\sec \beta + \tan \beta) - \sin \beta} . \]  
(11)

For the range \( \beta \leq \phi \leq \alpha \), it is found that the inequalities on \( n_\theta \) are satisfied
provided \( p \) satisfies the following two inequalities:

\[ 2 \sin^2 a \leq p \sin^2 \beta \leq 2 . \]  
(12)
However, the complexity of the expression (8) for $m_\phi$ precludes a general analytical discussion and hence $m_\phi$ was numerically evaluated and the corresponding inequalities verified. The range of validity of the solution [Eqs. (7), (8) and (9)] is indicated as region I in Fig. 7. For values of $\alpha$ and $\beta$ outside region I in Fig. 7 some other hypothesis must be made for the stress profile. The solution given in this paper is restricted to the values of $\alpha$ and $\beta$, which are consistent with the assumed stress profile. Since for the above region of validity, the stress profile is everywhere on regime 45, it follows from equations (5) that the velocity equations are
\[
\varepsilon_\phi = \dot{\theta} \cot \phi - \dot{\psi} = 0, \quad \varepsilon_\theta = \dot{\theta}' - \psi' = -\mu_\delta \\
x_\phi = -k \cot \phi (\dot{\psi} + \dot{\psi}') = \mu_\delta, \quad x_\theta = -k (\dot{\psi} + \dot{\psi}') = 0
\] (13)
where $\mu_\delta$ and $\mu_\delta$ are arbitrary positive multipliers. The solution of equations (13) which satisfies the boundary condition (6 b) is given by

![Fig. 7. Range of validity of the solution.](image)
Generalized Square Yield Condition for Shells of Revolution

\[ \dot{w} = \dot{w}_0 \cos \phi \left[ 1 - \frac{\log (\sec \phi + \tan \phi)}{\log (\sec \alpha + \tan \alpha)} \right] \]

\[ \dot{\phi} = \dot{w}_0 \sin \phi \left[ 1 - \frac{\log (\sec \phi + \tan \phi)}{\log (\sec \alpha + \tan \alpha)} \right] \]  

(14)

\[ \mu_4 = \frac{\tan \phi}{\log (\sec \alpha + \tan \alpha)} \]

\[ \mu_5 = k \dot{w}_0 \frac{\cot \phi}{\log (\sec \alpha + \tan \alpha)} \]  

(15)

where \( \dot{w}_0 \) is the velocity at the center of the shell. Hence \( \mu_4 \) and \( \mu_5 \) are always positive.

At \( \phi = 0 \), \( \nu = 0 \) and \( w' \) has a discontinuity in the proper direction for the hinge circle \( m_\phi = 1 \). At \( \phi = \alpha \), \( \nu = 0 \) so that the tangential displacement is continuous.

Therefore, the solution given by equations (7), (8), (9) and (14) satisfies the equations of equilibrium, the yield condition and associated flow rule and all the boundary conditions; further, it satisfies all stress inequalities and leads to positive multipliers for the strain-rate vector. Hence, it is the exact solution according to the generalized square-yield condition for the range of validity shown in Fig. 7.

The collapse pressure for a shell under uniform loading is obtained by setting \( \beta = \alpha \) in Eq. (9).

\[ [p]_{\beta=\alpha} = 2 + \frac{2k \sin \alpha}{[\log (\sec \alpha + \tan \alpha) - \sin \alpha]} \]  

(16)

The same value for the pressure was obtained by Hodge using a different yield condition.

Since \( P = pN_0/R \), Eq. (9) gives

\[ P = \frac{\left(\frac{2N_0}{R}\right) \left[ \log (\sec \alpha + \tan \alpha) - \sin \alpha \right] + k \sin \alpha}{\log (\sec \alpha + \tan \alpha) - \sin \beta - \cos^2 \beta \log \frac{\sec \alpha + \tan \alpha}{\sec \beta + \tan \beta}} \]  

(17)

The solution for the concentrated load is obtained by passing to the limit as the loaded area tends to zero. Since the collapse pressure is given by Eq. (17), the magnitude of the load at collapse is obtained by integrating Eq. (17) over the loaded area. Hence the collapse load \( F \) is given by
Substituting the expression for $P$ from Eq. (17) into Eq. (18) we get

$$F = 4\pi RN_0 \left[ \log (\sec a + \tan a) - (1 - k) \sin a \right] \left(1 - \cos \beta \right) \right].$$

By passing to the limit as $\beta$ tends to zero the concentrated collapse load is found to be

$$F_c = 2\pi RN_0 \left\{ 1 - (1 - k) \sin a \left[ \log (\sec a + \tan a) \right]^{-1} \right\}.$$

The velocity field given by Eqs. (14) and (15) remains unchanged as $\beta$ tends to zero.

The concentrated collapse load Eq. (20) can also be obtained by using the principle of virtual work. The rate at which mechanical energy (per unit area of the middle surface) is dissipated during the plastic flow is given by

$$di = N_o (n_\theta \epsilon_\theta + n_\phi \epsilon_\phi + m_\theta x_\theta + m_\phi x_\phi).$$

From the velocity field Eq. (14), we get

$$\epsilon_\theta = 0, \quad x_\phi = 0$$

and from the stress field Eqs. (7) and (8)

$$n_\phi = -1, \quad m_\theta = 1.$$ 

Hence Eq. (21) becomes

$$di = N_o (- \epsilon_\phi + x_\phi).$$

Substituting the expressions for the strain rates from Eqs. (13), (14) and (15) into Eq. (24) we get

$$di = \frac{N_0 w_o}{\log (\sec a + \tan a)} (\tan \phi + k \cot \phi).$$

The total internal rate of energy dissipation over the entire area of the shell surface is given by

$$D_i = \int_0^\alpha 2\pi R^2 di \sin \phi d\phi$$

$$= 2\pi R^2 N_0 w_o \left\{ 1 - (1 - k) \sin a \left[ \log (\sec a + \tan a) \right]^{-1} \right\}.$$

Since $w = W/R$, the total internal rate of energy dissipation can be written as
Generalized Square Yield Condition for Shells of Revolution

\[ D_t = 2\pi R N_0 W_0 \left(1 - (1 - k) \sin \alpha \left[ \log (\sec \alpha + \tan \alpha) \right]^{-1} \right). \] (27)

The total external rate of energy dissipation when the shell is subjected to a concentrated load \( F_c \) at the vertex is given by

\[ D_e = F_c W_0. \] (28)

Equating (27) and (28) we get back Eq. (20) for the concentrated collapse load. Figure 8 shows the collapse load \( F_c/2\pi R N_0 \) versus the angle of the cap \( \alpha \), for two values of \( k \). By allowing \( \alpha \) tend to zero in Eq. (20) the concentrated collapse load of a simply supported circular plate is obtained

\[ [F_c]_{\alpha \to 0} = 2\pi R N_0 k = 2\pi M_0. \] (29)

![Graph showing the relationship between collapse load and angle](image)

**Fig. 8.** Concentrated collapse load of a spherical cap.

The above result was originally obtained by Hopkins and Prager.\(^7\)

**SUMMARY**

A new yield condition is proposed for rotationally symmetric shells of revolution. Based on this yield condition the carrying capacity of a spherical cap under a concentrated load at the vertex is determined. It is found that the analysis based on the proposed yield condition offers considerable...
mathematical simplicity. It is hoped that the results obtained in this paper can be extended to the case of dynamic loading.

REFERENCES


