ON SOME FILTERED RINGS

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1. INTRODUCTION

O. ORE has defined a ring of non-commutative polynomials with coefficients in a field or skew-field K with respect to a derivation in K and has proved some decomposition theorems relating to the structure of such polynomials. These polynomials form a ring, also when the coefficient domain is an arbitrary ring Γ, with unit element. It is proved here that the class of these rings for various derivations are precisely the class of filtered rings whose associated graded rings are isomorphic to $\Gamma[X]$, the polynomial ring in one variable over Γ. $\Gamma[X]$ is a special case of the non-commutative polynomial ring when the derivation is zero. G. Hochschild has proved that the relative global dimension of $\Gamma[X]$ with respect to Γ is 1. This result is generalised here to the case, when the derivation is non-trivial.

2. THE CATEGORY $\mathcal{C}(\Gamma)$

A *filtered ring* is a ring Λ with an increasing sequence of subgroups $F_p\Lambda$, $p \in Z$ such that $F_p\Lambda . F_q\Lambda \subset F_{p+q}\Lambda$ and $\bigcup_{p \in Z} F_p\Lambda = \Lambda$. In what follows it is further assumed that $F_p\Lambda = 0$ for $p < 0$ and $F_0\Lambda = \Gamma$, a fixed ring with unit element. If Λ is a filtered ring, the graded group $E_0(\Lambda)$ whose p-th component is $F_p\Lambda/F_{p-1}\Lambda$ becomes a graded ring with the multiplication induced from Λ and this is called the *graded ring associated to Λ*. A *filtered left module* is a left Λ-module A with an increasing sequence of subgroups F_pA, $(p \geqslant 0)$ of A such that $A = \bigcup F_pA$ and $F_p\Lambda . F_qA \subset F_{p+q}A$ for all p, q. A homomorphism $\theta: \Lambda \to \Lambda'$ of filtered rings is a ring homomorphism such that $\theta(F_p\Lambda) \subset F_p\Lambda'$. θ_p denotes the restriction of θ to $F_p\Lambda$. θ clearly induces a homomorphism $E_0(\theta): E_0(\Lambda) \to E_0(\Lambda')$.

Consider the family of all pairs (Λ, ψ_Λ) where Λ is a filtered ring and $\psi_\Lambda: \Gamma[X] \to E_0(\Lambda)$ is an isomorphism of graded rings with $\psi_\Lambda{}^0 = \text{Identity}$ (*see* [4] Section 3). A map $\theta: (\Lambda, \psi_\Lambda) \to (\Lambda', \psi_\Lambda')$ is defined to be a ring homomorphism of Λ to Λ' compatible with the filtration such that the diagram

$$\begin{array}{ccc} E_0(\Lambda) & \xrightarrow{E_0(\theta)} & E_0(\Lambda') \\ & \nwarrow_{\psi_\Lambda} \quad {}^{\psi_{\Lambda'}}\nearrow & \\ & \Gamma[X] & \end{array}$$

is commutative. The composition of maps is defined in an obvious manner and the resulting category is denoted by $\mathcal{E}(\Gamma)$.

Lemma 1.—Every map in $\mathcal{E}(\Gamma)$ is in isomorphism.

Proof.—If

$$\theta: (\Lambda, \psi_\Lambda) \to (\Lambda', \psi_{\Lambda'})$$

is a map, then

$$E_0(\theta): E_0(\Lambda) \to E_0(\Lambda')$$

is an isomorphism since

$$E_0(\theta) = \psi_{\Lambda'} \circ \psi_\Lambda^{-1}.$$

Consider the following commutativity diagram

$$\begin{array}{ccccccccc} 0 & \to & F_{p-1}\Lambda & \to & F_p\Lambda & \to & E_0{}^p(\Lambda) & \to & 0 \\ & & \downarrow \theta_{p-1} & & \downarrow \theta_p & & \downarrow E_0{}^p(\theta) & & \\ 0 & \to & F_{p-1}\Lambda' & \to & F_p\Lambda' & \to & E_0{}^p(\Lambda') & \to & 0. \end{array}$$

Assume by induction that θ_{p-1} is an isomorphism. Since $E_0{}^p(\theta)$ is an isomorphism, θ_p is also an isomorphism.

3. Classification of Filtered Rings

In this section it is proved that the objects of $\mathcal{E}(\Gamma)$ are essentially the non-commutative polynomial rings of O. Ore[3] with respect to a suitable derivation of the base ring and conversely.

Proposition 1.—Let Γ be a ring with unit element. The isomorphism classes of objects in $\mathcal{E}(\Gamma)$ are in a (1, 1) correspondence with the group $H_Z{}^1(\Gamma, \Gamma)$ which is the group of derivations of Γ-modulo inner derivations.

Proof.—Let (Λ, ψ_Λ) be in $\mathcal{E}(\Gamma)$. The isomorphism $\psi_\Lambda{}^1: \Gamma_1[X] \to F_1\Lambda/\Gamma$ where $\Gamma_1[x]$ denotes the first homogeneous component of $\Gamma[X]$ induces an exact sequence of left Γ-modules

$$0 \to \Gamma \xrightarrow{i_\Lambda} F_1\Lambda \xrightarrow{\eta_\Lambda} \Gamma_1[X] \to 0$$

where η_Λ is a two-sided homomorphism since ψ_Λ is a ring homomorphism and $\psi_\Lambda{}^0 =$ Identity. Since $\Gamma_1[X]$ is a free left Γ-module, that exists a left Γ-homomorphism $t: \Gamma_1[X] \to F_1\Lambda$ such that $\eta_\Lambda o t =$ Identity. If $t(X) = X_1$, it follows that $\eta_\Lambda(X_1\gamma - \gamma X_1) = X\gamma - \gamma X = 0$ for each $\gamma \in \Gamma$. Thus there is a unique element of Γ which is denoted as $d\gamma$ such that $X_1\gamma - \gamma X_1 = d\gamma$. Clearly $d: \Gamma \to \Gamma$ is well defined and is a derivation. If $\underline{d}$ denotes the class of d in $H_z{}^1(\Gamma, \Gamma)$ $\underline{d}$ is independent of t. Indeed if $t': \Gamma_1[X] \to F_1\Lambda$ is another section with $t'(X) = X_1'$ and if $d'\gamma = X_1'\gamma - \gamma X_1'$ then $\eta(X_1' - X_1) = 0$. Hence $X_1' - X_1 = \omega \in \Gamma$ and clearly $d'\gamma - d\gamma = \omega\gamma - \gamma\omega$. Thus to each (Λ, ψ_Λ) in $\mathcal{E}(\Gamma)$ corresponds a well-defined element $\underline{d} \in H_z{}^1(\Gamma, \Gamma)$. It is easily seen that isomorphic objects in $\mathcal{E}(\Gamma)$ yield the same element of $H_z{}'(\Gamma, \Gamma)$.

Conversely any $\underline{d} \in H_z{}^1(\Gamma, \Gamma)$ gives rise to an isomorphism class of objects in $\mathcal{E}(\Gamma)$. Choose $d \in \underline{d}$ and let Λ_d be the free left Γ-module generated by

$$X_0 (= 1), X_1, \cdots, X_n, \cdots$$

with the multiplication defined by

$$X_n . X_m = X_{m+n}$$

$$X_n . \gamma = \sum_{i=0}^{n} \binom{\gamma}{i} d^i\gamma . X_{n-i} \qquad \gamma \in \Gamma$$

where $d^0: \Gamma \to \Gamma$ is the identity, d^i is the i-th iterate of d for $i > 0$ and the $\binom{r}{i}$ are the binomial coefficients. Λ_d is the non-commutative polynomial ring defined by O. Ore.[3] Λ_d becomes a filtered ring if $F_p\Lambda_d$ is defined such that it consists of all elements λ of the form

$$\lambda = \sum_{i \leqslant p} \gamma_i X_i, \quad \gamma_i \in \Gamma.$$

Consider the left Γ-homomorphism $\phi_d: \Gamma[X] \to \Lambda_d$ which is identity on Γ and maps X^n on X_n. This induces a mapping $\psi_d = E_0(\phi_d): \Gamma[X] \to E_0(\Lambda_d)$. It is easily verified that $\psi_d{}^0 =$ Identity and ψ_d is an isomorphism of graded rings. Thus (Λ_d, ψ_d) is an object in $\mathcal{E}(\Gamma)$. The map $\psi_d{}^1: \Gamma_1[X] \to F_1\Lambda_d/\Gamma$ gives an exact sequence

$$0 \to \Gamma \xrightarrow{i_d} F_1\Lambda_d \xrightarrow{\eta_d} \Gamma_1[X] \to 0.$$

Define $t: \Gamma_1[X] \to F_1\Lambda_d$ to be the restriction of ϕ_d. Since $t(X)\gamma - \gamma t(X) = X_1\gamma - \gamma X_1 = d\gamma$ it follows that (Λ_d, ψ_d) is mapped on to $\underline{d}$.

Finally if (Λ, ψ_Λ) is mapped on to $\underline{d}$, it is isomorphic to (Λ_d, ψ_d) for any $d \epsilon \underline{d}$. Since (Λ, ψ_Λ) is mapped on $\underline{d}$, for any $d \epsilon_d$, there exists d section

$$t_d : \Gamma_1 [X] \to F_1 \Lambda$$

of the exact sequence

$$0 \to \Gamma \to F_1 \Lambda \to \Gamma_1 [X] \to 0$$

such that $t_d (X) \gamma - \gamma t_d (X) = d\gamma$. t_d is extended to a Γ-linear map t_d: $\Gamma [X] \to \Lambda$ by defining it to be identity on Γ and such that X^n is mapped on to $\left(t_d (X)\right)^n$. On the other hand the Γ-linear map: $\Lambda_d \to \Gamma [X]$ which is identity on Γ and which maps X_n on X^n composed with t_d gives a Γ-linear map $\theta : \Lambda_d \to \Lambda$. θ is a ring homomorphism, since

$$\begin{aligned} \theta (X_n \cdot \gamma) &= \theta \left[\sum \binom{\gamma}{i} d^i \gamma \cdot X_{n-i} \right] \\ &= \sum \binom{\gamma}{i} d^i \gamma \cdot \left(t_d (X)\right)^{n-i} \\ &= (t_d (X)^n \cdot \gamma \\ &= \theta (X_n) \cdot \theta (\gamma), \quad \gamma \epsilon \Gamma. \end{aligned}$$

Further θ is compatible with the filtration and the diagram

$$\begin{array}{ccc} E_0 (\Lambda_d) & \xrightarrow{\;E_0 (\theta)\;} & E_0 (\Lambda) \\ & {}^{\psi_d}\nwarrow \qquad \nearrow^{\psi_\Lambda} & \\ & \Gamma [X] & \end{array}$$

is commutative. Thus $\theta : (\Lambda_d, \psi_d) \to (\Lambda, \psi_\Lambda)$ is a map in $\mathcal{C} (\Gamma)$ and hence an isomorphism from Lemma 1.

Proposition 2.—The isomorphisms of (Λ_d, ψ_d) on itself are in (1, 1) correspondence with $H_Z{}^0 (\Gamma, \Gamma) =$ centre of Γ.

Proof.—Let $\theta : (\Lambda_d, \psi_d) \to (\Lambda_d, \psi_d)$ be an isomorphism.

The diagram

$$\begin{array}{ccc} F_1 \Lambda_d / \Gamma & \xrightarrow{\;E_0{}^1 (\theta)\;} & F_1 \Lambda_d / \Gamma \\ & {}^{\psi_d{}^1}\nwarrow \qquad \nearrow^{\psi_d{}^1} & \\ & \Gamma_1 [X] & \end{array}$$

is commutative. If $\bar{X}_1 = \psi_{d}^{1}(X)$, clearly $\theta(X_1) = X_1 + \omega$ for some $\omega \epsilon \Gamma$.

$$\begin{aligned}\omega\gamma - \gamma\omega &= \big(\theta(X_1) - X_1\big)\gamma - \gamma\big(\theta(X_1) - X_1\big)\\ &= \theta(d\gamma) - d\gamma = 0.\end{aligned}$$

Hence ω is in the centre of Γ. Conversely if ω is in the centre of Γ, the Γ-linear map

$$\theta : \Lambda_d \to \Lambda_d$$

given by

$$\theta(X_1) = X_1 + \omega$$

is extendable to a ring homomorphism and is a map in $\mathcal{E}(\Gamma)$.

4. RELATIVE GLOBAL DIMENSION

In this section it is proved that the relative global dimension of Λ_d with respect to Γ is 1. Since Λ_d for $d = 0$ is the usual polynomial ring, this generalises a result of G. Hochschild.[2]

Lemma 2.—If X is a filtered complex of abelian groups such that the associated graded complex is acyclic, then X is acyclic.

Proof.—Assume by induction that the complex $F_{p-1}X$ is acyclic.

The exact sequence of complexes

$$0 \to F_{p-1}X \to F_pX \to E_0^p(X) \to 0$$

gives an exact homology triangle

$$\begin{array}{ccc} H\big(E_0^p(X)\big) & \longrightarrow & H(F_{p-1}X) \\ \nwarrow & & \nearrow \\ & H(F_pX) & \end{array}$$

Since $H(F_{p-1}X) = 0 = H\big(E_0^p(X)\big)$, it follows that $H(F_pX) = 0$. Thus F_pX is acyclic and since p is arbitrary $X = UF_pX$ is acyclic.

Proposition 3.—If (Λ, ψ_Λ) is in $\mathcal{E}(\Gamma)$, the relative global dimension of Λ with respect to Γ is 1.

Proof.—By proposition 1, Λ is isomorphic to Λ_d for some derivation d of Γ. Hence it is sufficient to prove the result for the ring Λ_d. Let $E(\gamma)$

$= E_0(y) \oplus E_1(y)$ be the Exterior algebra over Z in one variable y. For any left Λ_d-module M, consider the exact sequence

$$0 \rightarrow X^1 \xrightarrow{d_1} X^0 \xrightarrow{\mathscr{E}} M \rightarrow 0$$

where

$$X^1 = \Lambda_d \underset{\Gamma}{\otimes} M \underset{Z}{\otimes} E_1(y)$$

$$X^0 = \Lambda_d \underset{\Gamma}{\otimes} M \underset{Z}{\otimes} E_0(y)$$

$$d_1(r \otimes m \otimes y) = rX_1 \otimes m \otimes 1 - r \otimes X_1.m \otimes 1$$

$$\mathscr{E}(r \otimes m \otimes 1) = r.m \qquad r \in \Lambda_d, m \in M$$

clearly X^0, X^1 are (Λ_d, Γ)-projective and $\mathscr{E}, d_1$ are Λ_d-homomorphisms such that $\mathscr{E}_0 d_1 = 0$. The complex $X = X^0 \oplus X^1$ is filtered by setting

$$F_p X = F_{p-1}\Lambda_d \otimes M \otimes E_1(y) \oplus F_p\Lambda_d \otimes M \otimes E_0(y).$$

It is easily seen that $E_0{}^p(X)$ is isomorphic with

$$\Gamma_{p-1}[X] \otimes M \otimes E_1(y) \oplus \Gamma_p[X] \otimes M \otimes E_0(y).$$

Clearly $E_0{}^p(X)$ is the p-th homogeneous component of the complex established by G. Hochschild[2] for $d = 0$. Using the fact that the later complex is acyclic and Lemma 2, it follows that X is acyclic.

The map $h: M \rightarrow X^0$ given by $h(m) = 1 \otimes m \otimes 1$ is clearly a Γ-homotopy. Thus X is a left (Λ_d, Γ)-projective resolution of M. Hence the relative global dimension of Λ_d with respect to Γ is at most 1. Denote by Γ', the ring Γ converted into a left Λ_d-module by setting

$$\left(\sum_i a_i X_i\right) . \gamma = \sum_i a_i . d^i\gamma \qquad a_i, \gamma \in \Gamma.$$

A (Λ_d, Γ)-resolution of Γ' is given by

$$0 \rightarrow Y^1 \xrightarrow{d_1} Y^0 \xrightarrow{\mathscr{E}} \Gamma' \rightarrow 0$$

where Y^0, Y^1 are the same as X^0, X^1 with M replaced by Γ', and d_1, $\mathscr{E}$ the corresponding maps. A computation using this resolution gives

$$Ext^1_{(\Lambda_d, \Gamma)}(\Gamma', \Lambda_d) \simeq \frac{Hom_\Gamma(\Gamma', \Lambda_d)}{A}$$

where A consists of those homomorphisms f such that for some $g \in Hom_\Gamma(\Gamma', \Lambda_d)$, $f(\gamma) = X_1 g(\gamma) - g(d\gamma)$ for every $\gamma \in \Gamma$. The inclusion map

$i: \Gamma' \rightarrow A_d$ is not an element of A for otherwise $1 = i(1) = X_1 g(1) - g(d.1) = X_1 g(1)$, $g \in \mathrm{Hom}_{\Gamma}(\Gamma' A_d)$. This contradicts the fact that degree

$$X_1 g(1) > 0.$$

Thus

$$\mathrm{Ext}^1_{(A_d, \Gamma)}(\Gamma', A_d) \neq 0,$$

proving that the relative global dimension of A_d with respect to Γ is 1.

Remark.—A module of dimension 1 cannot be obtained (as in the case $d = 0$) from an arbitrary Γ-module M, since in the present case it is not possible to convert a Γ-module into a A_d-module by operating X trivially on M.

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5. REFERENCES

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