ON SOME FILTERED RINGS

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Received June 22, 1962

(Communicated by Dr. B. S. Madhava Rao, F.A.SC.)

1. INTRODUCTION

O. Ore has defined a ring of non-commutative polynomials with coefficients in a field or skew-field K with respect to a derivation in K and has proved some decomposition theorems relating to the structure of such polynomials. These polynomials form a ring, also when the coefficient domain is an arbitrary ring \( \Gamma \), with unit element. It is proved here that the class of these rings for various derivations are precisely the class of filtered rings whose associated graded rings are isomorphic to \( \Gamma [X] \), the polynomial ring in one variable over \( \Gamma \). \( \Gamma [X] \) is a special case of the non-commutative polynomial ring when the derivation is zero. G. Hochschild has proved that the relative global dimension of \( \Gamma [X] \) with respect to \( \Gamma \) is 1. This result is generalised here to the case, when the derivation is non-trivial.

2. THE CATEGORY \( \mathcal{G}(\Gamma) \)

A filtered ring is a ring \( A \) with an increasing sequence of subgroups \( F_p A, p \in \mathbb{Z} \) such that \( F_p A \cdot F_q A \subset F_{p+q} A \) and \( \bigcup_{p \in \mathbb{Z}} F_p A = A \). In what follows it is further assumed that \( F_p A = 0 \) for \( p < 0 \) and \( F_0 A = \Gamma \), a fixed ring with unit element. If \( A \) is a filtered ring, the graded group \( E_0 (A) \) whose \( p \)-th component is \( F_p A / F_{p-1} A \) becomes a graded ring with the multiplication induced from \( A \) and this is called the graded ring associated to \( A \). A filtered left module is a left \( A \)-module \( A \) with an increasing sequence of subgroups \( F_p A, (p \geq 0) \) of \( A \) such that \( A = \bigcup_{p \in \mathbb{Z}} F_p A \) and \( F_p A \cdot F_q A \subset F_{p+q} A \) for all \( p, q \). A homomorphism \( \theta : A \rightarrow A' \) of filtered rings is a ring homomorphism such that \( \theta (F_p A) \subset F_p A' \). \( \theta_p \) denotes the restriction of \( \theta \) to \( F_p A \). \( \theta \) clearly induces a homomorphism \( E_0 (\theta) : E_0 (A) \rightarrow E_0 (A') \).

Consider the family of all pairs \( (A, \psi_A) \) where \( A \) is a filtered ring and \( \psi_A : \Gamma [X] \rightarrow E_0 (A) \) is an isomorphism of graded rings with \( \psi_A^0 = \text{Identity} \) (see [4] Section 3). A map \( \theta : (A, \psi_A) \rightarrow (A', \psi_{A'}) \) is defined to be a ring homomorphism of \( A \) to \( A' \) compatible with the filtration such that the diagram
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$E_0(\Gamma) \xrightarrow{\theta} E_0(\Gamma')$

$\psi_\Delta \quad \quad \quad \psi_{\Delta'}$

$\Gamma [X]$

is commutative. The composition of maps is defined in an obvious manner and the resulting category is denoted by $\mathcal{C}(\Gamma)$.

**Lemma 1.**—Every map in $\mathcal{C}(\Gamma)$ is in isomorphism.

**Proof.**—If

$\theta : (A, \psi_A) \rightarrow (A', \psi_{A'})$

is a map, then

$E_0(\theta) : E_0(A) \rightarrow E_0(A')$

is an isomorphism since

$E_0(\theta) = \psi_{A'} \circ \psi_A^{-1}$.

Consider the following commutativity diagram

$0 \rightarrow F_{p-1} A \rightarrow F_p A \rightarrow E^P_0(A) \rightarrow 0$

$\downarrow \theta_{p-1} \quad \downarrow \theta_p \quad \downarrow E^P_0(\theta)$

$0 \rightarrow F_{p-1} A' \rightarrow F_p A' \rightarrow E^P_0(A') \rightarrow 0$.

Assume by induction that $\theta_{p-1}$ is an isomorphism. Since $E^P_0(\theta)$ is an isomorphism, $\theta_p$ is also an isomorphism.

3. **Classification of Filtered Rings**

In this section it is proved that the objects of $\mathcal{C}(\Gamma')$ are essentially the non-commutative polynomial rings of $O$. Ore with respect to a suitable derivation of the base ring and conversely.

**Proposition 1.**—Let $\Gamma$ be a ring with unit element. The isomorphism classes of objects in $\mathcal{C}(\Gamma)$ are in a (1,1) correspondence with the group $H_2^1(\Gamma, \Gamma)$ which is the group of derivations of $\Gamma$-modulo inner derivations.

**Proof.**—Let $(A, \psi_A)$ be in $\mathcal{C}(\Gamma)$. The isomorphism $\psi_A^1 : \Gamma_1 [X] \rightarrow F_1 A/\Gamma$ where $\Gamma_1 [x]$ denotes the first homogeneous component of $\Gamma [X]$ induces an exact sequence of left $\Gamma$-modules.
where $\eta_A$ is a two-sided homomorphism since $\psi_A$ is a ring homomorphism and $\psi_A^0 = \text{Identity}$. Since $\Gamma_1[X]$ is a free left $\Gamma$-module, that exists a left $\Gamma$-homomorphism $t: \Gamma_1[X] \rightarrow F_1 A$ such that $\eta_A ot = \text{Identity}$. If $t (X) = X_1$, it follows that $\eta_A (X_1 \gamma - \gamma X_1) = X_1 \gamma - \gamma X_1 = 0$ for each $\gamma \in \Gamma$. Thus there is a unique element of $\Gamma$ which is denoted as $d\gamma$ such that $X_1 \gamma - \gamma X_1 = d\gamma$.

Clearly $d : \Gamma \rightarrow \Gamma$ is well defined and is a derivation. If $d$ denotes the class of $d$ in $H_2^1(\Gamma, \Gamma)$ $d$ is independent of $t$. Indeed if $t' : \Gamma_1[X] \rightarrow F_1 A$ is another section with $t' (X) = X_1'$ and if $d' \gamma = X_1' \gamma - \gamma X_1'$ then $\eta (X_1' - X_1) = 0$.

Hence $X_1' - X_1 = \omega \epsilon \Gamma$ and clearly $d' \gamma - d\gamma = \omega \gamma - \gamma \omega$. Thus to each $(A, \psi_A)$ in $\xi (\Gamma)$ corresponds a well-defined element $d \epsilon H_2^1(\Gamma, \Gamma)$. It is easily seen that isomorphic objects in $\xi (\Gamma)$ yield the same element of $H_2^1(\Gamma, \Gamma)$.

Conversely any $d \epsilon H_2^1(\Gamma, \Gamma)$ gives rise to an isomorphism class of objects in $\xi (\Gamma)$. Choose $d \epsilon \mathcal{D}$ and let $A_d$ be the free left $\Gamma$-module generated by

$$X_0 (1), X_1, \cdots, X_n, \cdots$$

with the multiplication defined by

$$X_n \cdot X_m = X_{m+n}$$

$$X_n \cdot \gamma = \sum_{i=0}^{n} \binom{n}{i} d^i \gamma . X_{n-i} \quad \gamma \epsilon \Gamma$$

where $d^0 : \Gamma \rightarrow \Gamma$ is the identity, $d^i$ is the $i$-th iterate of $d$ for $i > 0$ and the $\binom{n}{i}$ are the binomial coefficients. $A_d$ is the non-commutative polynomial ring defined by O. Ore. $A_d$ becomes a filtered ring if $F_p A_d$ is defined such that it consists of all elements $\lambda$ of the form

$$\lambda = \sum_{i=0}^{p} \gamma_i X_i, \quad \gamma_i \epsilon \Gamma.$$ 

Consider the left $\Gamma$-homomorphism $\phi_d : \Gamma[X] \rightarrow A_d$ which is identity on $\Gamma$ and maps $X^n$ on $X_n$. This induces a mapping $\psi_d = E_0 (\phi_d) : \Gamma[X] \rightarrow E_0 (A_d)$. It is easily verified that $\psi_d^0 = \text{Identity}$ and $\psi_d$ is an isomorphism of graded rings. Thus $(A_d, \psi_d)$ is an object in $\xi (\Gamma)$. The map $\psi_d^1 : \Gamma_1[X] \rightarrow F_1 A_d / \Gamma$ gives an exact sequence

$$0 \rightarrow \Gamma \rightarrow F_1 A_d \rightarrow \Gamma_1[X] \rightarrow 0.$$ 

Define $t : \Gamma_1[X] \rightarrow F_1 A_d$ to be the restriction of $\phi_d$. Since $t (X) \gamma - \gamma t (X) = X_1 \gamma - \gamma X_1 = d\gamma$ it follows that $(A_d, \psi_d)$ is mapped on to $d$. 

0 \rightarrow \Gamma \rightarrow F_1 A_d \rightarrow \Gamma_1[X] \rightarrow 0.$$ 

Define $t : \Gamma_1[X] \rightarrow F_1 A_d$ to be the restriction of $\phi_d$. Since $t (X) \gamma - \gamma t (X) = X_1 \gamma - \gamma X_1 = d\gamma$ it follows that $(A_d, \psi_d)$ is mapped on to $d$. 

\[ 0 \rightarrow \Gamma \rightarrow F_1 A_d \rightarrow \Gamma_1[X] \rightarrow 0. \]
Finally if \((A, \psi_A)\) is mapped on to \(d\), it is isomorphic to \((A_d, \psi_d)\) for any \(d \in \mathcal{D}\). Since \((A, \psi_A)\) is mapped on \(d\), for any \(d \in \mathcal{D}\), there exists \(d\) section

\[ t_d: \Gamma_1 [X] \rightarrow F_1 A \]

of the exact sequence

\[ 0 \rightarrow \Gamma \rightarrow F_1 A \rightarrow \Gamma_1 [X] \rightarrow 0 \]

such that \(t_d(X) \gamma - \gamma t_d(X) = d\gamma\). \(t_d\) is extended to a \(\Gamma\)-linear map \(t_d: \Gamma [X] \rightarrow A\) by defining it to be identity on \(\Gamma\) and such that \(X^n\) is mapped on to \((t_d(X))^n\). On the other hand the \(\Gamma\)-linear map \(A_d \rightarrow \Gamma [X]\) which is identity on \(\Gamma\) and which maps \(X_n\) on \(X^n\) composed with \(t_d\) gives a \(\Gamma\)-linear map \(\theta: A_d \rightarrow A\). \(\theta\) is a ring homomorphism, since

\[
\theta(X_n \cdot \gamma) = \theta \left[ \sum \binom{n}{i} t_d(X)^i \cdot X_{n-i} \right]
\]

\[
= \sum \binom{n}{i} t_d(X)^i \cdot (t_d(X)^n)^{n-i}
\]

\[
= (t_d(X))^n \cdot \gamma
\]

\[
= \theta(X_n) \cdot \theta(\gamma), \quad \gamma \in \Gamma.
\]

Further \(\theta\) is compatible with the filtration and the diagram

\[
\begin{array}{ccc}
E_0(A_d) & \xrightarrow{E_0(\theta)} & E_0(A) \\
\downarrow \psi_d & & \downarrow \psi_A \\
\Gamma [X] & \xrightarrow{E_0(\theta)} & \Gamma [X]
\end{array}
\]

is commutative. Thus \(\theta: (A_d, \psi_d) \rightarrow (A, \psi_A)\) is a map in \(\mathcal{E}(\Gamma)\) and hence an isomorphism from Lemma 1.

_**Proposition 2.**_—The isomorphisms of \((A_d, \psi_d)\) on itself are in \((1, 1)\) correspondence with \(H_2^0(\Gamma, \Gamma) = \text{centre of } \Gamma\).

_Proof._—Let \(\theta: (A_d, \psi_d) \rightarrow (A_d, \psi_d)\) be an isomorphism.

The diagram

\[
\begin{array}{ccc}
F_1 A_d/\Gamma & \xrightarrow{E_0^{-1}(\theta)} & F_1 A_d/\Gamma \\
\downarrow \psi_d^{-1} & & \downarrow \psi_d^{-1} \\
\Gamma_1 [X] & & \Gamma_1 [X]
\end{array}
\]
is commutative. If $\tilde{X}_1 = \psi_d^1(X)$, clearly $\theta(X_1) = X_1 + \omega$ for some $\omega \in \Gamma$.

$$\omega \gamma - \gamma \omega = (\theta(X_1) - X_1) \gamma - \gamma (\theta(X_1) - X_1)$$

$$= \theta(dy) - dy = 0.$$

Hence $\omega$ is in the centre of $\Gamma$. Conversely if $\omega$ is in the centre of $\Gamma$, the $\Gamma$-linear map

$$\theta: A_d \to A_d$$

given by

$$\theta(X_1) = X_1 + \omega$$

is extendable to a ring homomorphism and is a map in $\xi(\Gamma)$.

4. Relative Global Dimension

In this section it is proved that the relative global dimension of $A_d$ with respect to $\Gamma$ is 1. Since $A_d$ for $d = 0$ is the usual polynomial ring, this generalises a result of G. Hochschild. 2

Lemma 2.—If $X$ is a filtered complex of abelian groups such that the associated graded complex is acyclic, then $X$ is acyclic.

Proof.—Assume by induction that the complex $F_{p-1}X$ is acyclic.

The exact sequence of complexes

$$0 \to F_{p-1}X \to F_pX \to E_0^p(X) \to 0$$

gives an exact homology triangle

$$\begin{array}{ccc}
H(E_0^p(X)) & \to & H(F_{p-1}X) \\
\downarrow & & \downarrow \\
H(F_pX) & & \\
\end{array}$$

Since $H(F_{p-1}X) = 0 = H(E_0^p(X))$, it follows that $H(F_pX) = 0$. Thus $F_pX$ is acyclic and since $p$ is arbitrary $X = UF_pX$ is acyclic.

Proposition 3.—If $(A, \psi_A)$ is in $\xi(\Gamma)$, the relative global dimension of $A$ with respect to $\Gamma$ is 1.

Proof.—By proposition 1, $A$ is isomorphic to $A_d$ for some derivation $d$ of $\Gamma$. Hence it is sufficient to prove the result for the ring $A_d$. Let $E(y)$
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Let $= E_0(y) \oplus E_1(y)$ be the Exterior algebra over $Z$ in one variable $y$. For any left $A_d$-module $M$, consider the exact sequence

$$0 \rightarrow X^1 \xrightarrow{d_1} X^0 \xrightarrow{\varphi} M \rightarrow 0$$

where

$$X^1 = A_d \otimes M \otimes E_1(y)$$

$$X^0 = A_d \otimes M \otimes E_0(y)$$

$$d_1 (r \otimes m \otimes y) = r X_1 \otimes m \otimes 1 - r \otimes X_1.m \otimes 1$$

$$\varphi (r \otimes m \otimes 1) = r.m \quad r \in A_d, m \in M$$

clearly $X^0, X^1$ are $(A_d, \Gamma)$-projective and $\varphi, d_1$ are $A_d$-homomorphisms such that $\varphi d_1 = 0$. The complex $X = X^0 \oplus X^1$ is filtered by setting

$$F_p X = F_{p-1} A_d \otimes M \otimes E_1(y) \oplus F_p A_d \otimes M \otimes E_0(y).$$

It is easily seen that $E_0^p (X)$ is isomorphic with

$$\Gamma_p [X] \otimes M \otimes E_1(y) \oplus \Gamma_p [X] \otimes M \otimes E_0(y).$$

Clearly $E_0^p (X)$ is the $p$-th homogeneous component of the complex established by G. Hochschild$^2$ for $d = 0$. Using the fact that the later complex is acyclic and Lemma 2, it follows that $X$ is acyclic.

The map $h: M \rightarrow X^0$ given by $h (m) = 1 \otimes m \otimes 1$ is clearly a $\Gamma$-homotopy. Thus $X$ is a left $(A_d, \Gamma)$-projective resolution of $M$. Hence the relative global dimension of $A_d$ with respect to $\Gamma$ is at most 1. Denote by $\Gamma'$, the ring $\Gamma$ converted into a left $A_d$-module by setting

$$(\Sigma a_i X_i). \gamma = \Sigma a_i . d^i \gamma \quad a_i, \gamma \in \Gamma'.$$

A $(A_d, \Gamma)$-resolution of $\Gamma'$ is given by

$$0 \rightarrow Y^1 \xrightarrow{d_1} Y^0 \xrightarrow{\varphi} \Gamma' \rightarrow 0$$

where $Y^0, Y^1$ are the same as $X^0, X^1$ with $M$ replaced by $\Gamma'$, and $d_1, \varphi$ the corresponding maps. A computation using this resolution gives

$$\text{Ext}^1_{(A_d, \Gamma)} (\Gamma', A_d) \simeq \frac{\text{Hom}_\Gamma (\Gamma', A_d)}{A}$$

where $A$ consists of those homomorphisms $f$ such that for some $g \in \text{Hom}_\Gamma (\Gamma', A_d), f(\gamma) = X_2 g (\gamma) - g (d_1 \gamma)$ for every $\gamma \in \Gamma$. The inclusion map
$i: \Gamma' \to \mathcal{A}_d$ is not an element of $A$ for otherwise $1 = i(1) = X_1 g(1) - g(d.1) = X_1 g(1), g \in \text{Hom}_\Gamma(\Gamma', \mathcal{A}_d)$. This contradicts the fact that degree $X_1 g(1) > 0$.

Thus

$$\text{Ext}^1(\mathcal{A}_d, \Gamma')(\Gamma', \mathcal{A}_d) \neq 0,$$

proving that the relative global dimension of $\mathcal{A}_d$ with respect to $\Gamma$ is 1.

Remark.—A module of dimension 1 cannot be obtained (as in the case $d = 0$) from an arbitrary $\Gamma$-module $M$, since in the present case it is not possible to convert a $\Gamma$-module into a $\mathcal{A}_d$-module by operating $X$ trivially on $M$.

I wish to express my thanks to Dr. R. Sridharan for his help in the preparation of this paper.

5. REFERENCES