THE ANISOTROPY OF ELASTIC WAVE PROPAGATION IN CRYSTALS

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I. Introduction

In an isotropic solid, there are three types of elastic wave-fronts moving in any direction, of which two are transverse and the third longitudinal. A crystal also admits three sets of wave-fronts in any direction inside it and the vibration directions of these wave-fronts are orthogonal to each other, but the vibration direction of each wave-front is generally obliquely inclined to its direction of propagation. With the exception of certain directions, which are determined by the symmetry of the lattice, elastic waves inside a crystal are not strictly longitudinal or transverse. It was shown in an earlier paper[1] that the maximum inclination of the vibration direction of a wave with its direction of propagation is of the order of ten to twelve degrees for Diamond, Magnesium oxide and Copper, which belong to the cubic class of symmetry. In the present paper these calculations have been extended to a few crystals belonging to other classes also. We may mention that this phenomenon of anisotropy in elastic wave propagation is strikingly manifested by the crystal Ammonium di-hydrogen phosphate which crystallizes in the tetragonal system, and for this substance the maximum inclination of the vibration direction with its direction of propagation exceeds slightly thirty-four degrees for the quasi-longitudinal branch.

Another problem of importance in elastic wave propagation concerns the form of the elastic wave-surface for solids. The wave-surface gives us an insight into the nature of the wave-fronts divulging from a point source inside a crystal, where a disturbance has been initially set up. It is generally defined as the envelope of all planes $(lx + my + nz - v) = 0$ where (l, m, n) denotes the direction of propagation of an elastic wave and v is the velocity in this direction. Since the velocity of a wave is a function of its direction, the wave surface for a crystal is not spherically symmetric. In view of the significant deviation which the vibration direction maintains with that of the direction of propagation for the crystal ADP, it was thought desirable to carry out calculations to determine the form of the wave-surface also for this crystal. These are reported in Section III and it is shown here that the direction of the ray which signifies the direction of transport of energy has a maximum inclination of twenty-eight degrees with its direction of propagation.

II. THE VIBRATION DIRECTIONS

If $\vec{A}$ (= Ax, Ay, Az) represents the amplitude of a plane elastic wave propagating in the direction (l, m, n) of a crystal, then the components of the displacement Ax, Ay and Az as well as the velocities of propagation of the three types of elastic waves are given by

$$R\vec{A} = 0 \tag{1}$$

where

$$R = \begin{bmatrix} A - \rho v^2 & \alpha\beta & \gamma\alpha \\ \alpha\beta & \beta - \rho v^2 & \beta\gamma \\ \gamma\alpha & \beta\gamma & C - \rho v^2 \end{bmatrix} \tag{1.a}$$

and

$$\begin{aligned}
A &= C_{11}l^2 + C_{66}m^2 + C_{55}n^2 + 2C_{56}mn + 2C_{15}ln + 2C_{16}lm \\
B &= C_{66}l^2 + C_{22}m^2 + C_{44}n^2 + 2C_{24}mn + 2C_{46}ln + 2C_{26}lm \\
C &= C_{55}l^2 + C_{44}m^2 + C_{33}n^2 + 2C_{34}mn + 2C_{35}ln + 2C_{45}lm \\
\beta\gamma &= C_{56}l^2 + C_{24}m^2 + C_{34}n^2 + (C_{23} + C_{44})\,mn + (C_{36} + C_{45})\,ln \\
&\quad + (C_{25} + C_{46})\,lm \\
\gamma\alpha &= C_{15}l^2 + C_{46}m^2 + C_{35}n^2 + (C_{36} + C_{45})\,mn + (C_{13} + C_{55})\,ln \\
&\quad + (C_{14} + C_{56})\,lm \\
\alpha\beta &= C_{16}l^2 + C_{26}m^2 + C_{45}n^2 + (C_{25} + C_{46})\,mn + (C_{14} + C_{56})\,ln \\
&\quad + (C_{12} + C_{66})\,lm.
\end{aligned} \tag{1.b}$$

The direction of the vector $\vec{A}$ is different from the direction (l, m, n) and in the sequel we study the inclination of these for a few crystals belonging to different classes of symmetry.

(i) *Hexagonal system.*—For the hexagonal system there are only five independent constants, namely, C_{11}, C_{33}, C_{44}, C_{12} and C_{13} and there exists relations of the type $C_{11} = C_{22}$; $C_{44} = C_{55}$; $C_{13} = C_{23}$; and $C_{66} = (C_{11} - C_{12})/2$.

All other elastic constants vanish. Applying these conditions, equation (1) reduces to

$$\begin{bmatrix} C_{11}l^2 + C_{66}m^2 + C_{44}n^2 - \rho v^2 & lm\,(C_{12} + C_{66}) & ln\,(C_{13} + C_{55}) \\ lm\,(C_{12} + C_{66}) & C_{66}l^2 + C_{11}m^2 + C_{44}n^2 - \rho v^2 & mn\,(C_{23} + C_{44}) \\ ln\,(C_{13} + C_{55}) & mn\,(C_{23} + C_{44}) & C_{55}l^2 + C_{44}m^2 + C_{33}n^2 - \rho v^2 \end{bmatrix} \vec{A} = 0. \tag{2}$$

Considering the wave propagation along the *xy* plane, *i.e.*, $n = 0$, we find that the waves become strictly longitudinal or transverse. Thus wave propagation is completely isotropic over the whole of the *xy* plane. This is obviously because of the relation $C_{66} = (C_{11} - C_{12})/2$ which is similar to the relation $C_{44} = (C_{11} - C_{12})/2$ that exists for isotropic solids. We may mention that of all the crystal classes, an isotropy in wave propagation over a whole plane exists for the hexagonal system only.

Passing the *yz* plane, the set of equations derived from (2) split into

$$\left.\begin{aligned} &Ax\left(\frac{C_{11} - C_{12}}{2} m^2 + C_{44}n^2 - \rho v^2\right) = 0 \\ &Ay\,(C_{11}m^2 + C_{44}n^2 - \rho v^2) + Az\,(C_{13} + C_{44})\,mn = 0 \\ &Ay\,(C_{13} + C_{44})\,mn + A_z\,(C_{44}m^2 + C_{33}n^2 - \rho v^2) = 0 \end{aligned}\right\}. \tag{3}$$

The wave-front having the velocity

$$V = \left\{\frac{\left(\frac{C_{11} - C_{12}}{2} m^2 + C_{44}n^2\right)}{\rho}\right\}^{\frac{1}{2}}$$

is transverse.

The other two values of v, which we denote by v_+ and v_-, give respectively the velocities of the quasi-longitudinal and quasi-transverse modes of vibration.

The angles which the vibrations directions of these two wave-fronts make with that of the direction of propagation $(0, m, n)$ are given by

$$\cos \phi_{\pm} = \frac{n}{D_{\pm}} [(C_{13} + C_{44})\, m^2 + (\rho v_{\pm}{}^2 - C_{11}m^2 - C_{44}n^2)] \qquad (4.a)$$

where

$$D_{\pm}{}^2 = (C_{13} + C_{44})^2\, m^2 n^2 + (\rho v_{\pm}{}^2 - C_{11}m^2 - C_{44}n^2)^2 \qquad (4.b)$$

The values of ϕ_- for the quasi-transverse mode as a function of θ, where θ is the angle between the wave-normal and y-axis, are plotted in Fig. 1 for the crystals Beryllium and β quartz. The graph of ϕ_+ for quasi-longitudinal mode *versus* θ is similar to the graph of ϕ_- *versus* θ since both the modes are mutually orthogonal. For Beryllium, the values of the elastic constants are based on the results of Gold,[2] *viz.*,

$$C_{11} = 30\cdot 8 \times 10^{11}; \quad C_{33} = 35\cdot 7 \times 10^{11}; \quad C_{44} = 11\cdot 0 \times 10^{11};$$
$$C_{12} = -\ 5\cdot 8 \times 10^{11}; \quad C_{13} = 8\cdot 7 \times 10^{11} \ \text{dynes/cm.}^2$$

For β quartz, the values are taken from those of Kammer, Pardue and Frissell,[3] *viz.*,

$$C_{11} = 11\cdot 66 \times 10^{11}; \quad C_{33} = 11\cdot 04 \times 10^{11}; \quad C_{44} = 3\cdot 606 \times 10^{11};$$
$$C_{12} = 1\cdot 67 \times 10^{11}; \quad C_{13} = 3\cdot 28 \times 10^{11} \ \text{dynes/cm.}^2$$

The maximum deviation of the wave-fronts from either longitudinality or transversality is very small for the crystal Beryllium and is of the order of 4° only; for β quartz, it is of the order of 10°.

(*ii*) *Tetragonal system.*—For the tetragonal system, equation (1) reduces to

$$\begin{bmatrix} C_{11}l^2 + C_{66}m^2 + C_{44}n^2 - \rho v^2 & lm\,(C_{12} + C_{66}) & ln\,(C_{13} + C_{44}) \\ lm\,(C_{12} + C_{66}) & C_{66}l^2 + C_{11}m^2 + C_{44}n^2 - \rho v^2 & mn\,(C_{13} + C_{44}) \\ ln\,(C_{13} + C_{44}) & mn\,(C_{13} + C_{44}) & C_{44}\,(l^2 + m^2) + C_{33}n^2 - \rho v^2 \end{bmatrix} \vec{A} = 0. \qquad (5)$$

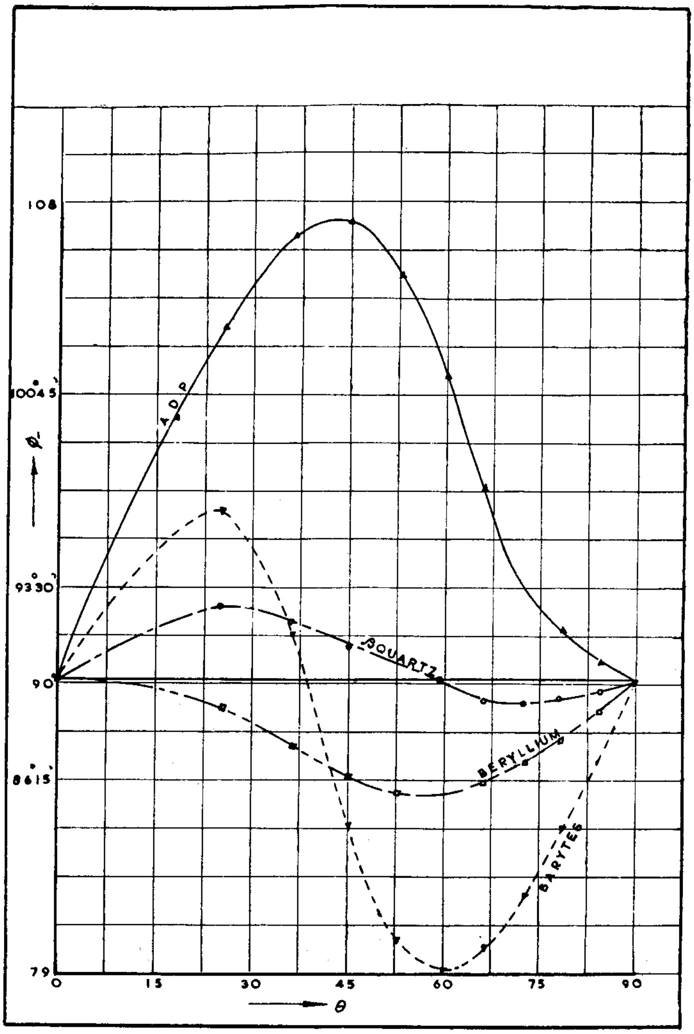

Fig. 1. Plot of ϕ_- as a function of θ in the case of ADP, Barytes, Beryllium and β quartz (ϕ'_- is the angle between the vibration direction of the wavefront of quasi-transverse mode with that of the wave-normal and θ is the angle between the wave-normal and y-axis.

Considering the propagation along the xy plane we get the equations,

$$\left.\begin{array}{l} Ax\,(C_{11}l^2 + C_{66}m^2 - \rho v^2) + Ay\,(C_{12} + C_{66})\,lm = 0 \\ Ax\,(C_{12} + C_{66})\,lm + Ay\,(C_{66}l^2 + C_{11}m^2 - \rho v^2) = 0 \\ Az\,\,(C_{44} - \rho v^2) = 0 \end{array}\right\} . \qquad (6)$$

The wave-front having the velocity $(C_{44}/\rho)^{\frac{1}{2}}$ is transverse. The inclinations of the other two wave-fronts with that of the wave-normal $(l, m, 0)$ are given by,

$$\cos \phi_{\pm} = \frac{m}{D_{\pm}} [l^2 (C_{12} + C_{66}) + (\rho v_{\pm}{}^2 - C_{11} l^2 - C_{66} m^2)] \qquad (7.a)$$

where

$$D_{\pm}{}^2 = (C_{12} + C_{66})^2 l^2 m^2 + (\rho v_{\pm}{}^2 - C_{11} l^2 - C_{66} m^2)^2 \qquad (7.b)$$

The deviations are calculated for the crystal Ammonium di-hydrogen phosphate. The values of $\phi_{\pm}$ as a function θ, where θ is the angle between the wave-normal and x-axis, are plotted in Figs. 2 and 3 respectively. The maximum values of ϕ_{+} and ϕ_{-} are 35° and 125° and so the deviations exhibited by this crystal are strikingly large. The steep fall of the curve at $\theta = 45°$ is surprising, and to understand this phenomenon the value of $(d\phi/d\theta)$ at $\theta = 45°$ was evaluated.

In fact

$$\left(\frac{d\phi}{d\theta}\right)_{\theta=45°} = \frac{C_{11} - C_{66}}{C_{12} + C_{66}} - 1.$$

This has a value of the order of 57 for ADP and gives a slope angle for the curve 89° approximately. The large value of the slope angle for ADP is due to the fact that C_{12} is negative and C_{66} is small and thus the denominator $(C_{12} + C_{66})$ in the expression for $(d\phi/d\theta)$ becomes very small. The slope of the curve at $\theta = 45°$ is not so pronounced for other crystals.

Turning to the yz plane we find that

$$\cos \phi_{\pm} = \frac{n}{D_{\pm}} [(C_{13} + C_{44})^2 m^2 + (\rho v_{\pm}{}^2 - C_{11} m^2 - C_{44} n^2)] \qquad (8.a)$$

where

$$D_{\pm}{}^2 = (C_{13} + C_{44})^2 m^2 n^2 + (\rho v_{\pm}{}^2 - C_{11} m^2 - C_{44} n^2)^2. \qquad (8.b)$$

For this plane the maximum inclinations are found to be 17° and 117° for the quasi-longitudinal branch and quasi-transverse branch respectively. Again the deviations are substantial. For the elastic constants, the following values given by Mason and Matthias[4] were used:

$C_{11} = 6\cdot2 \times 10^{11}$; $C_{33} = 3 \times 10^{11}$; $C_{44} = 0\cdot91 \times 10^{11}$;
$C_{66} = 0\cdot61 \times 10^{11}$; $C_{12} = -0\cdot5 \times 10^{11}$; $C_{13} = 1\cdot4 \times 10^{11}$
dynes/cm.^{2}

(*iii*) *Orthorhombic system.*—Finally for the orthorhombic system, the equation (1) simplifies to

$$\begin{bmatrix} C_{11}l^2 + C_{66}m^2 + C_{55}n^2 - \rho v^2 & lm\,(C_{12} + C_{66}) & ln\,(C_{13} + C_{55}) \\ lm\,(C_{12} + C_{66}) & C_{66}l^2 + C_{22}m^2 + C_{44}n^2 - \rho v^2 & mn\,(C_{23} + C_{44}) \\ ln\,(C_{55} + C_{13}) & mn\,(C_{23} + C_{44}) & (C_{55}l^2 + C_{44}m^2 + C_{33}n^2 - \rho v^2) \end{bmatrix} \vec{A} = 0. \tag{9}$$

For the xy plane, equation (9) reduces to:

$$\left.\begin{aligned} &Ax\,(C_{11}l^2 + C_{66}m^2 - \rho v^2) + Ay\,(C_{12} + C_{66})\,lm = 0 \\ &Ax\,(C_{12} + C_{66})\,lm + Ay\,(C_{66}l^2 + C_{22}m^2 - \rho v^2) = 0 \\ &Az\,(C_{55}l^2 + C_{44}m^2 - \rho v^2) = 0 \end{aligned}\right\}. \tag{10}$$

The wave-front advancing with the velocity

$$\left\{\frac{(C_{55}l^2 + C_{44}m^2)}{\rho}\right\}^{\frac{1}{2}}$$

is transverse. The deviations of the other two wave-fronts from the wave normal are given by

$$\cos\phi_{\pm} = \frac{m}{D_{\pm}}\left[(C_{12} + C_{66})\,l^2 + (\rho v_{\pm}{}^2 - C_{11}l^2 - C_{66}m^2)\right] \tag{11.a}$$

where

$$D_{\pm}{}^2 = (C_{12} + C_{66})^2\,l^2m^2 + (\rho v_{\pm}{}^2 - C_{11}l^2 - C_{66}m^2)^2. \tag{11.b}$$

The calculations have been carried out for Barytes using the values of the elastic constants given by Seshagiri Rao.[5] These values are

$C_{11} = 8\cdot62 \times 10^{11}$; $C_{22} = 9\cdot17 \times 10^{11}$; $C_{33} = 10\cdot84 \times 10^{11}$;
$C_{44} = 1\cdot2 \times 10^{11}$; $C_{55} = 2\cdot87 \times 10^{11}$; $C_{66} = 2\cdot74 \times 10^{11}$;
$C_{12} = 5\cdot23 \times 10^{11}$; $C_{13} = 3\cdot41 \times 10^{11}$; $C_{23} = 3\cdot56 \times 10^{11}$
dynes/cm.^{2}

The values of ϕ_+ and ϕ_- are plotted against θ in Figs. 2 and 3 respectively. Their maximum values are found to be 4° 30′ and 93° while their mean values lie about 3° and 91° respectively. The calculations have been extended to the other two planes, *i.e.*, *yz* and *zx* for which the angular deviations are respectively given by the equations

$$\cos \phi_{\pm} = \frac{n}{D_{\pm}} [(C_{23} + C_{44}) m^2 + (\rho v_{\pm}{}^2 - C_{22}m^2 - C_{44}n^2)] \quad (12.a)$$

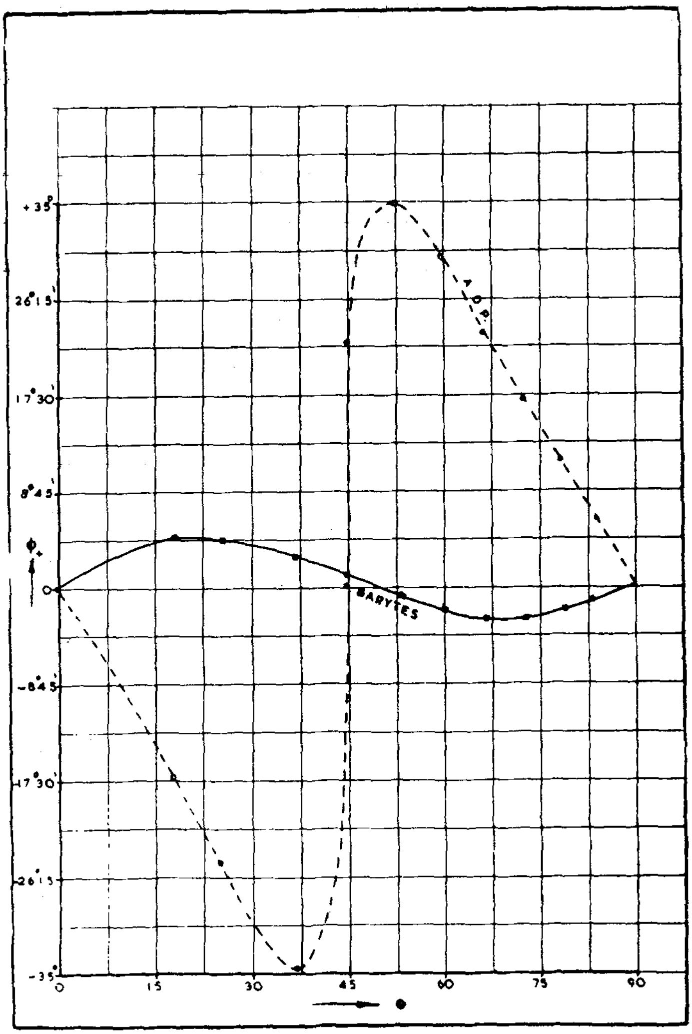

FIG. 2. Plot of ϕ_+ as a function of θ for ADP and Barytes ϕ_+ is the angle between the vibration direction of quasi-longitudinal mode and wave normal. θ is the angle between wave-normal and x-axis.

where

$$D_{\pm}{}^2 = (C_{23} + C_{44})^2 m^2n^2 + (\rho v_{\pm}{}^2 - C_{22}m^2 - C_{44}n^2)^2 \quad (12.b)$$

and

$$\cos \phi_{\pm} = \frac{n}{D_{\pm}} [(C_{13} + C_{55}) l^2 + (\rho v_{\pm}^2 - C_{11} l^2 - C_{55} n^2)] \qquad (13.a)$$

where

$$D_{\pm}^2 = (C_{13} + C_{55})^2 l^2 n^2 + (\rho v_{\pm}^2 - C_{11} l^2 - C_{55} n^2)^2 \qquad (13.b)$$

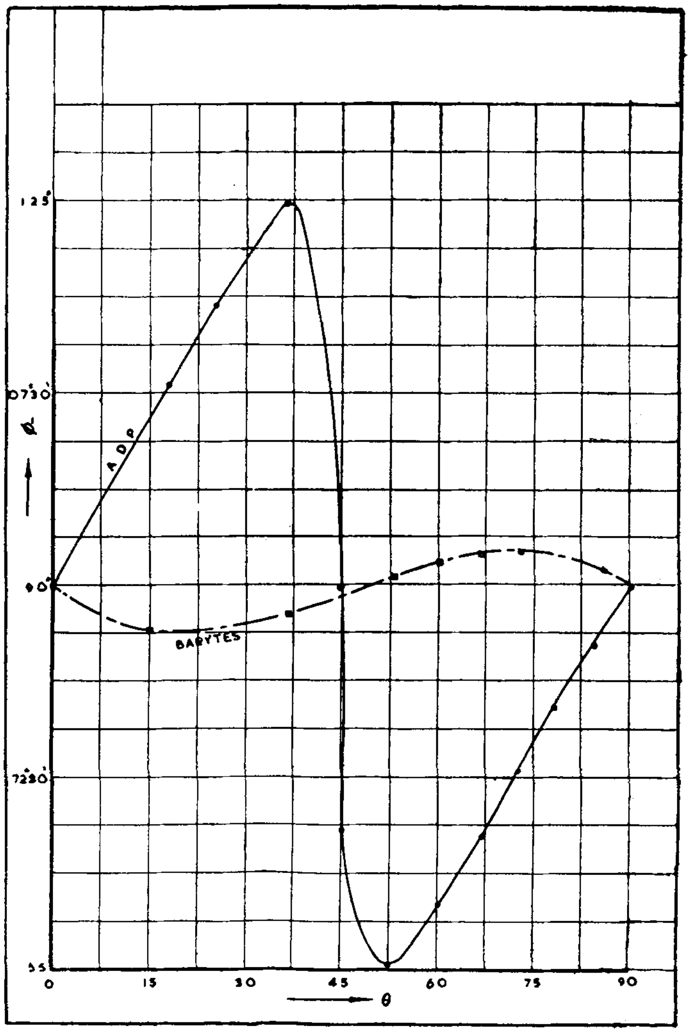

FIG. 3. Plot of ϕ_- as a function of θ for ADP and Barytes. ϕ_- is the angle between the vibration direction of quasi-transverse mode and wave-normal. θ is the angle between wave-normal and x-axis.

In the case of yz plane, the deviation is of the order of 10° and it is plotted in Fig. 1. For zx plane it is of the order of 6°.

III. Wave-Surface

The wave-surface is the envelope of planes of the form

$$l\xi + m\eta + n\zeta - v = 0 \tag{14}$$

where (ξ, η, ζ) is a point on the wave-surface and v the phase velocity. Further l, m, n and v are subject to the conditions

$$l^2 + m^2 + n^2 = 1. \tag{15}$$

$$\frac{a^2}{\rho v^2 - A + a^2} + \frac{\beta^2}{\rho v^2 - B + \beta^2} + \frac{\gamma^2}{\rho v^2 - C + \gamma^2} = 1. \tag{16}$$

Solving the equation (14) with the subsidiary conditions (15) and (16), we finally arrive at the following relations due to Musgrave[6]

$$v - A' = \frac{1}{2\rho v}[lL + mM + nN]$$

$$\xi = l\left[v - \frac{1}{2\rho v}(lL + mM + nN)\right] + \frac{L}{2\rho v}$$

$$\eta = m\left[v - \frac{1}{2\rho v}(lL + mM + nN)\right] + \frac{M}{2\rho v}$$

$$\zeta = n\left[v - \frac{1}{2\rho v}(lL + mM + nN)\right] + \frac{N}{2\rho v} \tag{17}$$

where A′ is an undetermined multiplier and L, M and N are given by

$$L = \left(\frac{p^2}{a^2}\right)A_l + \left(\frac{q^2}{\beta^2}\right)B_l + \left(\frac{r^2}{\gamma^2}\right)\Gamma_l$$

$$A_l = \frac{\partial a^2}{\partial l}(\rho v^2 - A) + a^2\frac{\partial A}{\partial l}$$

$$B_l = \frac{\partial \beta^2}{\partial l}(\rho v^2 - B) + \beta^2\frac{\partial B}{\partial l}$$

$$\Gamma_l = \frac{\partial \gamma^2}{\partial l}(\rho v^2 - c) + \gamma^2\frac{\partial c}{\partial l}$$

$$p:q:r = \frac{Ax:Ay:Az}{(Ax^2 + Ay^2 + Az^2)^{\frac{1}{2}}}. \tag{18}$$

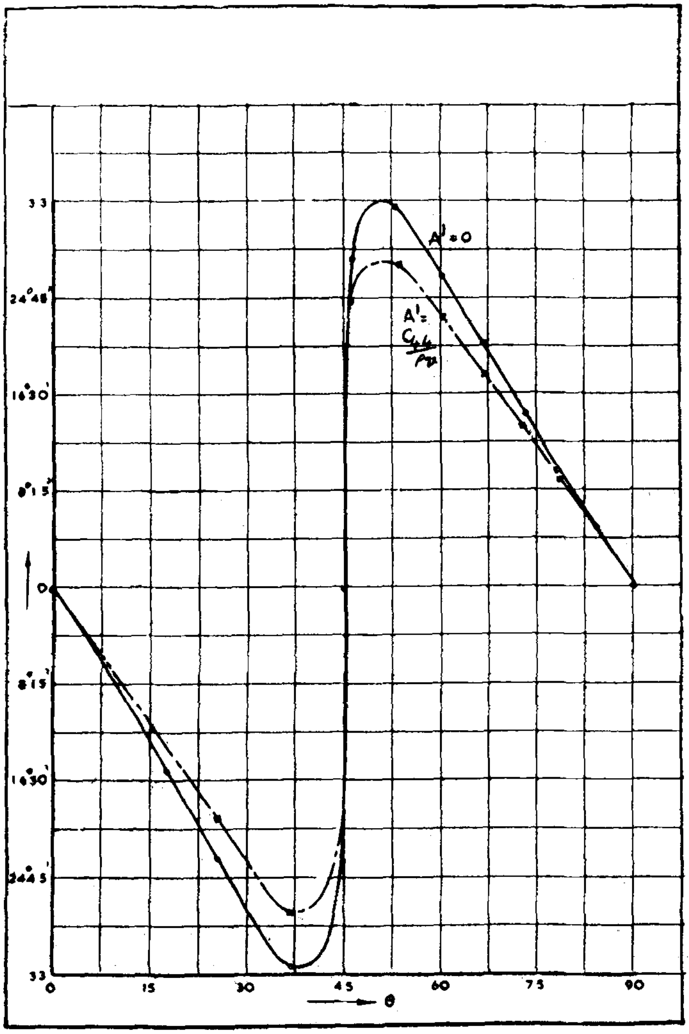

FIG. 4. Plot of ψ for xy plane as a function of θ for ADP for two values of A', A' = 0 and A' = $C_{44}/\rho v$. ψ is the angle between ray vector and wave-normal: θ is the angle between wave-normal and x-axis. A' is an arbitrary constant.

A, B, C, α, β and γ are given by the equation (1.b).

We have two similar equations for M and N.

The values of ξ, η, ζ are determined for the two cases, *viz.*, (*i*) A' = 0 and (*ii*) A' = $C_{44}/\rho v$.

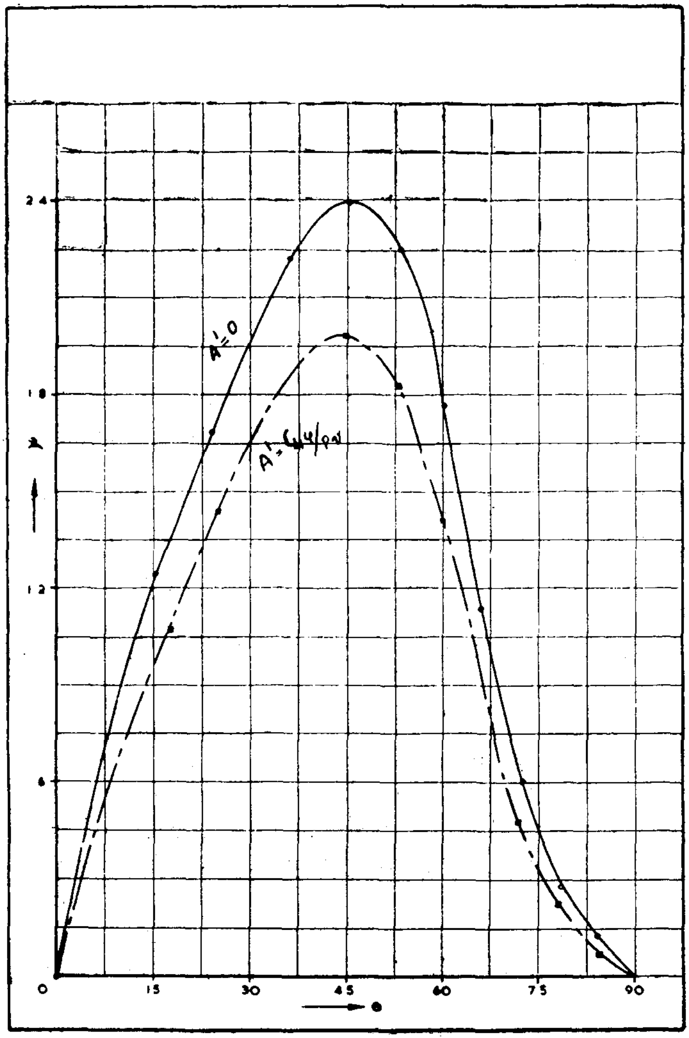

FIG. 5. Plot of ψ for yz plane as a function of θ for ADP and Barytes for two values of A', A' = 0 and A' = $(C_{44}/\rho v)$; ψ is the angle between the ray vector and wave-normal; θ is the angle between wave-normal and y-axis; A' is an arbitrary constant.

The co-ordinates ξ, η, ζ in these two cases are given by

$$(i)\ \xi = \frac{L}{2\rho v}\ ;\quad \eta = \frac{M}{2\rho v}\ ;\quad \zeta = \frac{N}{2\rho v} \tag{19}$$

$$(ii)\ \xi = \frac{2C_{44}l + L}{2\rho v}\ ;\quad \eta = \frac{2C_{44}m + M}{2\rho v}\ ;\quad \zeta = \frac{2C_{44}n + N}{2\rho v} \tag{20}$$

Further, the angles between the ray vector and wave-normal are given by the equations,

$$(i)\ \cos\psi = \frac{l\xi + m\eta + \eta\zeta}{(\xi^2 + \eta^2 + \zeta^2)^{\frac{1}{2}}} = \frac{lL + mM + nN}{(L^2 + M^2 + N^2)^{\frac{1}{2}}} \tag{21}$$

$$(ii)\ \cos\psi = \frac{2C_{44} + lL + mM + nN}{\{(2C_{44}l + L)^2 + (2C_{44}m + M)^2 + (2C_{44}n + N)^2)\}^{\frac{1}{2}}}. \tag{22}$$

For the two cases, the calculations were carried out for Ammonium di-hydrogen phosphate for xy and yz planes and the calculated values for ψ are shown in Figs. 4 and 5. The maximum values for ψ in the case $A' = 0$ are of the order of 33° and 24° respectively for the two planes xy and yz. In the case $A' = (C_{44}/\rho v)$ the maximum inclinations are of the order of 28° and 20° respectively. As in the case of the inclination of the displacement vector with the wave-normal for xy plane, here also, the inclination of the ray vector changes suddenly from maximum to minimum near the point $\theta = 45°$.

Finally I express my grateful thanks to Dr. K. S. Viswanathan for his guidance and to Dr. P. Nilakantan, Director, National Aeronautical Laboratory, for his valuable criticisms and kind encouragement.

IV. Summary

A numerical estimation of the angles between wave-normal and vibration direction of the elastic waves for various classes of crystal has been given. The anisotropy in elastic wave propagation is strikingly manifested by the crystal Ammonium di-hydrogen phosphate belonging to the tetragonal system, which shows a maximum inclination of thirty-four degrees. Numerical studies have also been carried out for this crystal to obtain an estimate of the deviation of the ray vector from the direction of the wave-normal.

V. References

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