ON THE EXISTENCE AND UNIQUENESS OF FLOWS BEHIND THREE-DIMENSIONAL CURVED STOCKS

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INTRODUCTION

By using Cauchy-Kowaleski theory, Taub discussed the existence and uniqueness of flows behind two-dimensional stationary and pseudo-stationary curved shocks. Later Kanwal generalised his results for flows behind three-dimensional stationary and pseudo-stationary shocks. In their treatment, the existence theorem is applied after reducing the basic equations of gas dynamics to the ‘normal’ form by using the formal methods of tensor analysis. It is shown in the present paper that it is simpler to apply Cauchy-Kowaleski theory directly to the equations without first transforming them. The same method is used for discussing the conditions for the existence and uniqueness of flows behind unsteady curved shocks as well as flows behind curved shocks in three-dimensional magnetogas-dynamics.

THE BASIC EQUATIONS AND SHOCKS CONDITIONS

The shock surface divides the flow into two regions 1 (in front of the shock) and 2 (behind the shock). For the case of an ideal stationary gas with viscosity and heat conductivity zero, the following equations hold on both sides of the shock surface:

\[
\begin{align*}
\rho_i, iu_i + \rho u_j, j & = 0 \quad \text{(equation of continuity)} \\
p, i + \rho u_j u_i, j & = 0 \quad \text{(equation of motion)} \\
u_S, i & = 0 \quad \text{(equation of energy)} \\
p - \rho \gamma', e^{s,JC}, \gamma & = 0 \quad \text{(equation of state)}
\end{align*}
\]

where \( p, \rho, S \) and \( u_i \) denote the pressure, density, entropy and velocity components respectively. From (3) and (4) we get on eliminating \( S \),

\[
u_i, j u_i u_j - c^2 u_k, k = 0, \quad c^2 = \frac{\gamma p}{\rho}.
\]
Let $u_{ai}$, $p_1$, $p_1$, and $u_{ai}$, $p_2$, $p_2$ denote the values of the variables immediately in front of and behind the shock surface, then the Rankine-Hugoniot conditions for the stationary shock are:

$$u_{ai} - u_{ai} = -\frac{\delta}{1+\delta} n_i u_{in}$$
$$p_2 - p_1 = \frac{\delta p_1 u_{in}^2}{1+\delta}$$
$$\rho_2 - \rho_1 = \delta \rho_1,$$  \hspace{1cm} (6)

where $n_i$ are the components of the normal to the shock surface directed from region 1 into region 2, and

$$u_{in} = u_{i1n_i}, \quad u_{2n} = u_{i2n_i}$$ \hspace{1cm} (7)

$$\delta = \frac{2 (\rho_1 u_{in}^2 - \gamma p_1)}{2\gamma p_1 + (\gamma - 1) \rho_1 u_{in}^2}.$$ \hspace{1cm} (8)

**Existence and Uniqueness of Flows behind Three-Dimensional Stationary Curved Shocks**

Omitting the suffix 2 for the quantities behind the shock, the five partial differential equations of the first order for the five variables $u_1$, $u_2$, $u_3$; $p$, $\rho$ are given by (1), (2) and (5). Knowing the flow in front of the shock and the equation of the shock surface, equations (6) determine the values of these variables on the shock surface for the flow in region 2. The Cauchy-Kowalewski condition for the existence of the flow behind the shock is:

$$\rho n_1 \begin{vmatrix} \rho n_2 & \rho n_3 & 0 & u_1 n_1 + u_2 n_2 + u_3 n_3 \\ (u_1 n_1 + u_2 n_2 + u_3 n_3) & 0 & 0 & n_1 \\ 0 & \rho (u_1 n_1 + u_2 n_2 + u_3 n_3) & 0 & n_2 \\ (u_1 n_1 + u_2 n_2 + u_3 n_3) & 0 & \rho (u_1 n_1 + u_2 n_2 + u_3 n_3) & n_3 \end{vmatrix} \neq 0$$ \hspace{1cm} (9)

or

$$\rho u_n^4 \left(1 - \frac{c^2}{u_n^2}\right) \neq 0,$$ \hspace{1cm} (10)

but

$$1 - \frac{c^2}{u_n^2} = -\frac{\gamma + 1}{2} \delta.$$  \hspace{1cm} (11)

Since $\delta > 0$ for a shock of non-zero strength, the required condition is
satisfied and there exists a unique analytic flow behind a steady curved shock. We may note that (9) is the same condition as obtained in [1] and [2].

EXISTENCE AND UNIQUENESS OF FLOWS BEHIND THREE-DIMENSIONAL NON-STATIONARY CURVED SHOCKS

The basic equations in this case are:

\[ \rho, t + u_j \rho, j + \rho u_j, j = 0 \]  
(11)

\[ \rho u_t, t + \rho, i + \rho u_j u_i, j = 0 \]  
(12)

\[ S, t + u_i S, i = 0 \]  
(13)

From (4) and (13)

\[ p, t + u_j p, j = c^2 (\rho, t + u_j \rho, j) \]  
(14)

Using (11), (12) and (14)

\[ -\rho^{-1} \rho, t + u_i u_i, t + u_i u_j u_i, j - c^2 u_j, j = 0. \]  
(15)

Equations (11), (12) and (15) constitute the five partial differential equations of the first order for determining the five functions \( u_1, u_2, u_3, \rho, p \) of the four independent variables \( x_1, x_2, x_3, t \).

Let the equation of the shock surface be

\[ s (x_1, x_2, x_3, t) = 0. \]  
(16)

For the flow behind to exist and be unique, the determinant \( \triangle \) should not vanish, where

\[ \triangle = \begin{vmatrix}
\rho s_{x_1} & \rho s_{x_2} & \rho s_{x_3} & 0 & u_1 s_{x_1} + u_2 s_{x_2} + u_3 s_{x_3} + s, t \\
\rho [u_4 s_{x_1} + u_2 s_{x_2} + u_3 s_{x_3} + s, t] & 0 & 0 & s_{x_1} & 0 \\
0 & \rho [u_4 s_{x_1} + u_2 s_{x_2} + u_3 s_{x_3} + s, t] & 0 & s_{x_2} & 0 \\
0 & 0 & \rho [u_4 s_{x_1} + u_2 s_{x_2} + u_3 s_{x_3} + s, t] & s_{x_3} & 0 \\
(u_1^2 - c^2) s_{x_1} & u_2 u_4 s_{x_1} + (u_2^2 + c^2) s_{x_2} & u_3 u_4 s_{x_1} + u_2 u_4 s_{x_2} + \frac{1}{\rho} s, t & 0 & 0 \\
+ u_1 u_2 s_{x_2} & + u_1 u_2 s_{x_3} & + u_2 u_3 s_{x_2} + s, t & 0 & 0 \\
+ u_1 u_3 s_{x_3} & + u_1 u_3 s_{x_1} & + u_2 s, t & 0 & 0 \\
+ u_1 s, t & + u_2 s, t & + u_3 s, t & + u_4 s, t & 0 \\
\end{vmatrix} \]  
(17)
Existence & Uniqueness of Flows behind Three-Dimensional Curved Stocks

Let

\[ n_i = \frac{s_i t}{\sqrt{s_{i1}^2 + s_{i2}^2 + s_{i3}^2}} (i = 1, 2, 3), \quad n_\delta = \frac{s_\delta t}{\sqrt{s_{\delta1}^2 + s_{\delta2}^2 + s_{\delta3}^2}} \]  

(18)

and

\[ u_n = u_1 n_1 + u_2 n_2 + u_3 n_3. \]  

(19)

then the condition becomes on simplification

\[ (u_n + n_\delta)^3 \left\{ (u_n + n_\delta)^2 - c^2 \right\} \neq 0. \]  

(20)

Now at the shock surface if \( v_1, v_2, v_3 \) are the velocity components for the shock, then

\[ s_1 v_1 + s_2 v_2 + s_3 v_3 + s, t = 0 \]

or

\[ n_1 v_1 + n_2 v_2 + n_3 v_3 + n_\delta = 0, \]  

(21)

so that

\[ u_n + n_\delta = u_n - n_i v_i = n_i u_2 - n_i v_i \]

\[ = n_i (u_i - v_i) = n_i U_i = U_n \quad \text{(say)}. \]  

(22)

Thus the condition is given by

\[ 1 - \frac{c^2}{U_n^2} = - \frac{\gamma + 1}{2} \delta \neq 0 \]  

(23)

which is satisfied in the case of a shock of non-zero strength. The present result generalises the results of \(^{[1]}\) and \(^{[2]}\) for pseudo-stationary shocks and shows that even in the most general case, the result obtained there holds.

**Condition for the Existence of Flows behind Three-Dimensional Curved Shocks in Magneto-Gasdynamics**

Assuming zero viscosity, zero heat conductivity and infinite electrical conductivity, the basic equations of magneto-gasdynamics to be satisfied on both sides of the shock are:

\[ u_j H_i, j - H_j u_i, j + H_j u_j, j = 0 \]  

(24)

\[ u_j u_i, j - \frac{\mu e}{\rho} H_j H_i, j + \frac{1}{\rho} (p, i + \mu e H_j H_j, i) = 0 \]  

(25)

\[ u_j p, j + \rho u_j, j = 0 \]  

(26)

\[ u_j p, j - c^2 u_j p, j = 0. \]  

(27)
These are eight partial differential equations for the eight variables \( u_1, u_2, u_3, H_1, H_2, H_3, p, \rho \). The values of these immediately in front are to be determined by generalised Rankine-Hugoniot relations. From Cauchy-Kowaleski theory the flow will exist behind the shock if the following eighth order determinant \( \Delta' \) does not vanish.

\[
\begin{vmatrix}
-H_n & H_1 n_2 & H_1 n_3 & u_n & 0 & 0 & 0 & 0 \\
+H_2 n_2 & -H_n & H_2 n_3 & 0 & u_n & 0 & 0 & 0 \\
+H_3 n_3 & H_3 n_2 & -H_n & 0 & 0 & u_n & 0 & 0 \\
\rho / \mu e & H_n & 0 & 0 & H_1 n_1 & H_2 n_1 & H_3 n_1 & 1 / \mu e \\
0 & u_n & 0 & 0 & H_1 n_2 & H_2 n_2 & H_3 n_2 & 1 / \mu e \\
0 & 0 & u_n & 0 & H_1 n_3 & H_2 n_3 & H_3 n_3 & 1 / \mu e \\
n_1 & n_2 & n_3 & 0 & 0 & 0 & 0 & u_n / \rho \\
0 & 0 & 0 & 0 & -u_n & c^2 u_n & \neq 0 \quad (28)
\end{vmatrix}
\]

where \( H_n \) and \( u_n \) are the normal components of the magnetic and velocity fields immediately behind the shock. For two-dimensional flows, this reduces to a sixth order determinant. Its particular cases as well as the corresponding condition for non-steady case will be discussed separately. The aim above is to show that the direct method is very simple to apply even in the more general case of magneto-gasdynamics. It need hardly be pointed out that the method of Taub and Kanwal is not at all applicable in this case.

**REFERENCES**


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