SOME GENERAL KERNELS FOR THE DERIVATION OF SELF-RECIPIROCAL FUNCTIONS

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INTRODUCTION

The aim of this paper is to give some new kernels and deduce some self-reciprocal functions under the Hankel transform.

2. A function \( f(x) \) is said to be self-reciprocal under Hankel's transform and is denoted by \( R_\nu \) if the following integral equation is satisfied:

\[
f(y) = \int_0^\infty f(x) J_\nu(xy) \, dx
\]

where \( J_\nu(x) \) is the Bessel function of order \( \nu \). Titchmarsh\(^1\) gives that \( f(x) \) is \( R_\nu \), if

\[
f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{S/\nu} \Gamma\left(\frac{S}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \psi(S) x^{-S} \, dS \quad 0 < c < 1
\]

where \( \psi(S) \) satisfies the functional relation

\[
\psi(S) = \psi(1 - S).
\]

In other words \( f(x) \) \( R_\nu \) is if its Mellin transform is of the form

\[
2^{S/\nu} \Gamma\left(\frac{S}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \psi(S)
\]

where \( \psi(S) \) satisfies (2.3)

Again if \( f(x) \) is \( R_\mu \) and

\[
g(x) = \int_0^\infty f(y) h(xy) \, dy
\]

is \( R_\nu \), then \( h(x) \) is called a kernel which carries \( R_\mu \) into \( R_\nu \).
Titchmarsh\textsuperscript{1} gives that if

\[ h(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^s \Gamma\left(\frac{S}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \Gamma\left(\frac{S}{2} + \frac{\mu}{2} + \frac{1}{4}\right) \phi(S) x^{-s} \, ds \]

where

\[ \phi(S) = \phi(1 - S) \quad (2.5) \]

then \( h(x) \) is a kernel which carries \( R_\mu \) into \( R_\nu \). The symmetry of (2.5) in \( \mu \) and \( \nu \) suggests that \( h(x) \) transforms \( R_\nu \) into \( R_\mu \) also. In other words if the Mellin transform of \( h(x) \) is

\[ 2^s \Gamma\left(\frac{S}{2} + \frac{\mu}{2} + \frac{1}{4}\right) \Gamma\left(\frac{S}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \phi(S) \]

where \( \phi(S) \) satisfies (2.6), then \( h(x) \) is a kernel which carries \( R_\mu \) into \( R_\nu \) and \textit{vice versa}.

3. It is now proposed to prove that

\[ x^{1/2} K_{\mu/2-n/2} (\alpha x) J_{\mu/2+n/2} (\beta x) \]

is a kernel which carries \( R_\mu \) into \( R_\nu \) provided \( \alpha^2 + \beta^2 = 1 \).

Watson\textsuperscript{2} gives that

\[ \int_0^\infty K_{\mu/2-n/2} (\alpha t) J_{\mu/2+n/2} (\beta t) t^{-\rho} \, dt = \frac{\beta^\rho \Gamma\left(\frac{\nu - \rho + \mu + 1}{2}\right) \Gamma\left(\frac{\nu - \rho - \mu + 1}{2}\right)}{2^{\rho+1} \alpha^{\rho+1} \Gamma(\nu+1)} \times _2F_1\left(\frac{\nu - \rho + \mu + 1}{2}, \frac{\nu - \rho - \mu + 1}{2}; \nu + 1; -\frac{\beta^2}{\alpha^2}\right). \]

\[ \text{Re}(\alpha \pm i\beta) > 0, \quad \text{Re}(\nu + \rho + 1 \pm \mu) > 0 \quad (3.1) \]

Taking \( -\rho = \lambda - 1 \), we have that

\[ \int_0^\infty K_{\mu/2-n/2} (\alpha t) J_{\mu/2+n/2} (\beta t) t^{\lambda-1} \, dt = \frac{\beta^\lambda \Gamma\left(\frac{\nu + \mu + \lambda}{2}\right) \Gamma\left(\frac{\nu - \mu + \lambda}{2}\right)}{2^{\lambda-\lambda} \alpha^{\lambda} \Gamma(\nu+1)} \times _2F_1\left(\frac{\nu + \mu + \lambda}{2}, \frac{\nu - \mu + \lambda}{2}; \nu + 1; -\frac{\beta^2}{\alpha^2}\right) \]

\[ \text{Re}(\alpha \pm i\beta) > 0, \quad \text{Re}(\nu + \lambda \pm \rho) > 0 \quad (3.2) \]
By writing the Hypergeometric series $F(a, b; c; z)$ as

$$(1 - z)^{-a} F\left(a, c - b; c; \frac{z}{z - 1}\right)$$

assuming that $|z| < \frac{1}{2}$ we have from (3.2) that

$$\int_0^\infty K_{\mu}(at) J_{\nu}(\beta t) t^{\lambda - 1} dt$$

$$= \frac{\beta^\nu \Gamma\left(\frac{\nu + \mu + \lambda}{2}\right) \Gamma\left(\frac{\nu - \mu + \lambda}{2}\right)}{2^{2-\lambda} a^{\nu-\lambda} \Gamma(\nu + 1)} \left(\frac{\alpha^2}{\alpha^2 + \beta^2}\right)^{(\nu + \mu + \lambda)/2}$$

$$\times _2F_1\left(\frac{\nu + \mu + \lambda}{2}, \frac{\nu - \mu + \lambda}{2}; \nu + 1; \frac{\beta^2}{\alpha^2 + \beta^2}\right). \quad (3.3)$$

From (3.3), we get that the Mellin transform of

$$x^{1/2} K_{\mu}(ax) J_{\nu}(\beta x)$$

is

$$C 2^\lambda \Gamma\left(\frac{\lambda}{2} + \frac{\nu + \mu + 1}{2}\right) \Gamma\left(\frac{\lambda}{2} + \frac{\nu - \mu + 1}{2}\right)$$

$$\times _2F_1\left(\frac{\nu + \mu + \lambda}{2}, \frac{\nu - \mu + \lambda + 3}{4}; \nu + 1; \beta^2\right) \quad (3.4)$$

provided $\alpha^2 + \beta^2 = 1$ where $C$ is a constant not involving $\lambda$ which satisfies conditions laid down in §2.

Hence

$$x^{1/2} K_{\mu}(ax) J_{\nu}(\beta x)$$

is a kernel which carries $R_{\nu-\mu}$ into $R_{\mu-\nu}$ and vice versa provided $\alpha^2 + \beta^2 = 1$.

Taking $\nu + \mu = \nu, \nu - \mu = \nu$ we have that

$$x^{1/2} K_{(\mu-\nu)/2}(ax) J_{(\mu+\nu)/2}(\beta x)$$

is a kernel which carries $R_{\mu}$ into $R_{\nu}$ and vice versa provided $\alpha^2 + \beta^2 = 1$ and $\text{Re} (\mu, \nu) > -1$.

We now give the particular cases. If $\beta \to 0$ then $a \to 1$ and we get that

$$x^{(\mu+\nu+1)/2} K_{(\mu-\nu)/2}(x)$$

is a kernel which carries $R_{\mu}$ into $R_{\nu}$ and vice versa as given by Titchmarsh.
Taking
\[ \alpha = \beta = \frac{1}{\sqrt{2}}, \quad \nu = 0 \]
we have that
\[ x^{1/2} J_{\mu} \left( \frac{x}{\sqrt{2}} \right) K_{\mu} \left( \frac{x}{\sqrt{2}} \right) \]
is a kernel which carries \( R_0 \) into \( R_{\mu} \) and \textit{vice versa} as given by Gupta.\(^3\)

4. Similarly we prove that
\[ x^{1/2} J_{(\mu+\nu)/2} (ax) J_{(\mu-\nu)/2} (bx) \]
is a kernel which carries \( R_{\mu} \) into \( R_{\nu} \) and \textit{vice versa} provided \( \alpha^2 - \beta^2 = 1 \). Watson\(^2\) gives that
\[
\int_0^\infty J_{\mu} (at) J_{\nu} (bt) t^{-\rho} \, dt
\]
\[
= \frac{\alpha^\mu \Gamma \left( \frac{\nu + \mu - \rho + 1}{2} \right)}{2^\rho \beta^{\mu-\rho+\lambda} \Gamma (\mu + 1) \Gamma \left( 1 + \frac{\nu - \rho + 1}{2} \right)}
\times \, _2F_1 \left( \frac{\nu + \mu - \rho + 1}{2}, \frac{\mu - \nu - \rho + 1}{2}; \mu + 1; \frac{a^2}{\beta^2} \right)
\]
\[ \text{Re} (\nu + \mu - \rho + 1) > 0, \quad \text{Re} \rho > -1, \quad 0 < \alpha < \beta. \quad (4.1) \]

(4.1) can be rewritten as
\[
\int_0^\infty J_{\mu} (at) J_{\nu} (bt) t^{\lambda-1} \, dt
\]
\[
= \frac{\alpha^{\mu} \Gamma \left( \frac{\nu + \mu + \lambda}{2} \right)}{2^{1-\lambda} \Gamma (\mu + 1) \Gamma \left( 1 + \frac{\nu - \mu - \lambda}{2} \right) \beta^{\mu+\lambda}}
\times \, _2F_1 \left( \frac{\nu + \mu + \lambda}{2}, \frac{\mu - \nu + \lambda}{2}; \mu + 1; \frac{a^2}{\beta^2} \right)
\]
\[ \text{Re} (\nu + \mu + \lambda) > 0 \quad \text{and} \quad \lambda > 2 \quad (4.2) \]

proceeding similarly to §3 we get that the Mellin transform of
\[ x^{1/2} J_{(\mu+\nu)/2} (ax) J_{(\mu-\nu)/2} (bx) \]
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is

\[ C 2^S \Gamma \left( \frac{S}{2} + \frac{\beta^2}{4} + \frac{1}{4} \right) \Gamma \left( \frac{S}{2} + \frac{\beta^2}{4} + \frac{1}{4} \right) \times \frac{\Gamma \left( \frac{S}{2} + \frac{1}{4}, \frac{\beta^2}{2} - \frac{3}{4}; \frac{\beta^2}{2} + \frac{1}{4}; \alpha^2 \right)}{\Gamma \left( \frac{\beta^2}{2} + \frac{1}{4}, \frac{1}{2}; \frac{\beta^2}{2} - \frac{3}{4}; \frac{\beta^2}{2} + \frac{1}{4}; \alpha^2 \right)} \]

provided \( \alpha^2 - \beta^2 = 1 \) and \( |\alpha^2/\beta^2| < \frac{1}{2} \). where \( C \) is a constant not involving \( S \).

Hence we get that

\[ x^{1/2} J_{(\mu - \nu)/2} (\alpha x) J_{(\mu - \nu)/2} (\beta x) \]

is kernel which carries \( R_\mu \) into \( R_\nu \) and \textit{vice versa} provided \( \alpha^2 - \beta^2 = 1 \) and \( \text{Re} (\mu, \nu) > -1 \).

Particular case: Making \( \beta \to 0 \) we get that

\[ x^{(\mu - \nu + 1)/2} J_{(\mu + \nu)/2} (x) \]

is a kernel which carries \( R_\mu \) into \( R_\nu \) and \textit{vice versa} given by Titchmarsh.\(^1\)

5. In § 3 we have proved that

\[ x^{1/2} K_{(\mu - \nu)/2} (\alpha x) I_{(\mu + \nu)/2} (\beta x) \]

is a kernel which carries \( R_\mu \) into \( R_\nu \) and \textit{vice versa} provided \( \alpha^2 + \beta^2 = 1 \). Taking \( \beta = i\beta \) and using the relation

\[ J_{\mu} (ix) = i^\mu I_{\mu} (x) \]

we get that

\[ x^{1/2} i^{(\mu + \nu)/2} K_{(\mu - \nu)/2} (\alpha x) I_{(\mu + \nu)/2} (\beta x) \]

is a kernel which carries \( R_\mu \) into \( R_\nu \) provided \( \alpha^2 - \beta^2 = 1 \). Leaving the constant factor we get that

\[ K_{(\mu - \nu)/2} (\alpha x) I_{(\mu + \nu)/2} (\beta x) x^{1/2} \]

is a kernel which carries \( R_\mu \) into \( R_\nu \) and \textit{vice versa} provided \( \alpha^2 - \beta^2 = 1 \) and \( \text{Re} (\mu, \nu) > -1 \).

Particular case: Making \( \beta \to 0 \) we get the well known result that

\[ x^{(\mu + \nu + 1)/2} K_{(\mu + \nu)/2} (x) \]

is a kernel which carries \( R_\mu \) into \( R_\nu \).

6. We now use the kernels given above on some of the known self-reciprocal functions and deduce new self-reciprocal functions. \( x^{1/2} e^{-x^2/2} \) is \( R_0 \) and from § 4 we have that

\[ x^{1/2} e^{-x^2/2} \]
\( f^{1/2} J_\mu (\alpha t) J_\mu (\beta t) \)

is a kernel which carries \( R_0 \) into \( R_{2\mu} \) provided \( \alpha^2 - \beta^2 = 1 \). Hence

\[
g(x) = \int_0^{\infty} f^{1/2} e^{-t^{1/2}} (\alpha xt) J_\mu (\alpha xt) J_\mu (\beta xt) \, dt
\]

(6.1)
is \( R_{2\mu} \). By 13.31 Watson we have that

\[
y^{1/2} \exp \left\{ -\frac{1}{2} + 2a^2 y^2 \right\} \, I_\mu \{a \sqrt{1 + a^2 y^2}\} \text{ is } R_{2\mu}.
\]

(6.2)

If \( \alpha \to 0 \) we get that \( x^{2\mu + 1/2} e^{-x^s/2} \) is \( R_{2\mu} \).

We know that

\[
x^{1/2} J_{(\mu + \nu)/2}(\alpha x) J_{(\mu - \nu)/2}(\beta x)
\]
is a kernel which carries \( R_\mu \) into \( R_\nu \) provided \( \alpha^2 - \beta^2 = 1 \). Erdelyi and others give that

\[
\int_0^{\infty} J_\mu (\alpha t) J_\nu (\beta t) e^{-\lambda t^s} \, t^{s-1} \, dt
\]

\[
= \frac{\alpha^\mu \beta^\nu}{2^{\mu+\nu} \lambda^{\mu+\nu}} \sum_{m=0}^{\infty} \frac{\Gamma(S + \mu + \nu + 2m)}{m! \Gamma(\mu + m + 1)} \left( \frac{-\alpha}{4\lambda^s} \right)^m
\]

\[
\times \quad _2 F_1 \left( -m, -\mu - m; \nu + 1; \frac{\beta^3}{\alpha^2} \right)
\]

(6.3)

\( \text{Re } (S + \mu + \nu) > 0 \).

By making suitable adjustments we get that

\[
\int_0^{\infty} (\alpha xt)^{1/2} J_{(\mu + \nu)/2}(\alpha xt) J_{(\mu - \nu)/2}(\beta xt) e^{-t^{1/2}} \, t^{s+1/2} \, dt
\]

\[
= x^{\mu + 1/2} \sum_{m=0}^{\infty} \frac{\Gamma(\mu + \nu + 2m + 2)}{m! \Gamma\left( \frac{\mu + 2m + 2}{2} \right)} \left( \frac{-a^2 x^s}{2} \right)^m
\]

\[
\times \quad _2 F_1 \left( -m, -\mu - m; \nu + 1; \frac{\beta^3}{\alpha^2} \right).
\]

(6.4)
gives an \( R_\mu \) function provided \( \alpha^2 - \beta^2 = 1 \). When \( \beta^2 = 0 \) we get from (6.4) \( x^{\mu + 1/2} e^{-x^s/2} \) as a particular case.

Using the above kernel on

\[
x^{\nu + 2\nu + 1/2} e^{-x^s/2} L_\nu^{\nu + 1} \left( \frac{x^s}{2} \right)
\]
which is $R_\mu$ for integral values of $n$ we get on using the integral (6.3) after simplification that

$$\int^{\mu+1/2}_0 \sum_{m=0}^n \sum_{k=0}^\infty \frac{(-1)^m 2^{m/2} \Gamma \left(\frac{2K + 2n + 3\mu + m - \nu + 2}{2}\right)}{m!K! \Gamma \left(\frac{2K + \mu - \nu + 2}{2}\right)} \left(-\frac{a^2 x^2}{2}\right)^k F_1 \left(-K, -\frac{2K + \mu + \nu}{2}; \frac{\nu - \mu + 2}{2}; \frac{\beta^2}{a^2}\right)$$

is $R_\mu$ provided $a^2 - \beta^2 = 1$.

7. Erdelyie and others\(^5\) give that the Mellin transform of $e^{-ax} J_{\mu}(\beta x)$ is

$$\frac{\beta^\nu \Gamma(S + \nu)}{2^\nu \alpha^{\nu + 8} \Gamma(\nu + 1)} F_1 \left(\frac{S + \nu}{2}, \frac{S + \nu + 1}{2}; \nu + 1; -\frac{\beta^2}{\alpha^2}\right)$$

$$\text{Re } \alpha > |\text{Im } \beta| \text{ Re } S > -\text{ Re } \nu.$$  \hspace{1cm} (7.1)

By effecting suitable changes we have that

$$t^{1/2} e^{-at} J_{\nu/2}(\beta t^2)$$

has

$$C \cdot 2^{5/2} \Gamma \left(\frac{\nu + S + 1}{2}\right) F_1 \left(\frac{S + \nu + 1}{2}, S - 1; \frac{\nu - S + 3}{8}; \frac{\beta^2}{a^2}\right)$$

as its Mellin transform provided $a^2 + \beta^2 = 1$ where $C$ is a constant. Hence by §3 we have that $t^{1/2} e^{-at} J_{\nu/2}(\beta t^2)$ is $R_\mu$ provided $a^2 + \beta^2 = \frac{1}{4}$. Making $\beta \rightarrow 0$ we get that $t^{\nu + 1/2} e^{-t^2/2}$ is $R_\mu$ which is a well-known result (7.2).

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