A FEW SELF-RECIPROCAL FUNCTIONS

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1. The object of this note is to find out a few Self-Reciprocal Functions for Hankel transforms. I denote a function \( f(x) \) as \( R_\mu \), if it is Self-Reciprocal for Hankel transforms of order \( \mu \) so that it is given by

\[
f(x) = \int_0^\infty J_\mu(xy)f(y) \sqrt{xy} \, dy ,
\]

where \( J_\mu(x) \) is a Bessel function of order \( \mu \). If \( \mu = \frac{1}{2} \), then \( f(x) \) is denoted by \( R_s \), while if it is \( R - \frac{1}{2} \), it is written as \( R_c \).

Hardy and Titchmarsh\(^6\) have shown that under certain broad conditions \( f(x) \) is \( R_\mu \), if

\[
f(x) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} 2^{si/2} \Gamma \left( \frac{1}{4} + \frac{\mu}{2} + \frac{s}{2} \right) \psi(s) x^{-s} \, ds ,
\]

where \( 0 < c < 1 \), and \( \psi(s) = \psi(1 - s) \).

If \( \psi(s) = -\psi(1 - s) \), then \( f(x) \) is denoted by \( -R_\mu \) and in that case \( f(x) \) is said to be Skew reciprocal for Hankel transforms of order \( \mu \).

2. We start with the integral given by Watson\(^12\)

\[
\int_0^\infty \frac{J_{a+p}(ax) J_{a-p-1}(ax)}{x^\lambda} \, dx = \frac{a^{\lambda-1} \Gamma \left( \frac{\lambda}{2} + \frac{1}{2} \right) \Gamma \left( a - \frac{\lambda}{2} \right)}{2 \Gamma \left( p + \frac{\lambda}{2} + 1 \right) \Gamma \left( a + \frac{\lambda}{2} \right) \Gamma \left( \frac{\lambda}{2} - p \right) }
\]

where \( \lambda \neq 0 \).
Putting $\lambda = (1 - s)$, $d = 3/2$, $p = -\frac{1}{4}$ and $a = 1/\sqrt{2}$, we obtain that

$$\int_0^\infty \left[ J_1 \left( \frac{x}{\sqrt{2}} \right) \right]^2 t^{s-1} dt = \frac{2^{s/2}}{2} \cdot \frac{1}{\Gamma \left( 2 - \frac{s}{2} \right)},$$

(2.2)

where $s \neq 1$.

\[ \therefore \text{ Applying Mellin's Inversion formula} \]

we obtain

$$\left[ J_1 \left( \frac{x}{\sqrt{2}} \right) \right]^2 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s/2} \Gamma \left( \frac{3}{2} + \frac{s}{2} \right) \frac{\Gamma \left( 2 - \frac{s}{2} \right)}{\Gamma \left( \frac{3}{2} + \frac{s}{2} \right)} x^{-s} ds,

= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s/2} \Gamma \left( \frac{3}{2} + \frac{s}{2} \right) \psi(s) x^{-s} ds,$$

(2.3)

where $0 < c < 1$, and $\psi(s) = \psi(1 - s)$.

Hence from (2.3) and (1.2) we find that

$$\left[ J_1 \left( \frac{x}{\sqrt{2}} \right) \right]^2$$

(2.4)

is $R_{5/2}$.

3. Goldstein\(^a\) has shown that

$$\int_0^\infty t^{l-1} e^{t/2} I_\nu \left( 2a\sqrt{t} \right) W_{k,m} (t) dt = \frac{\Gamma \left( \frac{\nu}{2} + l + m + \frac{1}{2} \right) \Gamma \left( \frac{\nu}{2} + l - m + \frac{1}{2} \right)}{\Gamma (\nu + 1) \Gamma \left( \frac{\nu}{2} + l - k + 1 \right)}$$

$$\times a^{\nu} \text{F}_2 \left[ \begin{array}{c} \frac{\nu}{2} + l + m + \frac{1}{2}, \frac{\nu}{2} + l - m + \frac{1}{2} \\ a^2, \nu + 1, \frac{\nu}{2} l - k + 1 \end{array} \right]$$

(3.1)

where $\left( l + \frac{\nu}{2} - m + \frac{1}{2} \right) > 0$.\(^a\)
Putting

\[ k = \frac{p}{2} - \frac{5}{4}, \quad m = \frac{p}{2} - \frac{1}{4}, \quad \frac{t^2}{2} \text{ for } t \text{ and } \frac{a}{\sqrt{2}} \text{ for } a, \]

we find that (3.1), becomes

\[
\int_0^\infty (at)^{2l-1} e^{at/4} W_{\frac{p}{2}-\frac{3}{4}, \frac{1}{4}} \left( \frac{t^2}{2} \right) I_v(at) \, dt
\]

\[
= \frac{\Gamma\left( \frac{v}{2} + l + \frac{p}{2} + \frac{1}{4} \right) \Gamma\left( \frac{v}{2} + l - \frac{p}{2} + \frac{3}{4} \right) a^{p+q-1}}{\Gamma(v - 1) \Gamma\left( \frac{v}{2} + l - \frac{p}{2} + \frac{3}{4} \right) 2^{p-1}}
\]

\[ \times \, _2F_2 \left[ \begin{array}{c} \frac{v}{2} + l + \frac{p}{2} + \frac{1}{4}, \frac{v}{2} + l - \frac{p}{2} + \frac{3}{4} \\ \frac{a^2}{2}, \nu + 1, \nu + l - \frac{p}{2} + \frac{9}{4} \end{array} \right] \]

where \( \left( l + \frac{v}{2} - \frac{p}{2} + \frac{3}{4} \right) > 0. \quad (3.2) \]

Bailey\(^a\) has shown that

\[ e^{t^2/4} W_{\frac{p}{2}-\frac{3}{4}, \frac{1}{4}} \left( \frac{t^2}{2} \right) \quad (3.3) \]

is \( R_p \).

Also Dr. Brij Mohan\(^4\) has shown that the kernel

\[ x^{1/2(p+q+1)} I_{1/2(p-q)}(x), \quad (3.4) \]

transforms \( R_p \) into \( R_q \).

Hence putting

\[
\frac{1}{2} (p + q + 1) = 2l - 1,
\]

and

\[
\frac{1}{2} (p - q) = \nu,
\]

we find that

\[
p = 2l + \nu - \frac{3}{2},
\]

and

\[
q = 2l - \nu - \frac{3}{2}
\]

Applying the results of (3.5), (3.4), and (3.3) to (3.2) we conclude that

\[ x^{2p+\mu+1/2} _2F_2 \left[ \begin{array}{c} 2\nu + \mu + 1, \frac{3}{2}; \, \frac{x^2}{2} \\ \nu + 1, 3 \end{array} \right] \quad (3.6) \]

is \( R_\mu \).
Some of the particular cases of this function are interesting.

Putting \( \nu = 1 - \frac{\mu}{2} \), we find that

\[
x^{3/3} \, _1F_1\left( 2 - \frac{\mu}{2}; \frac{3}{2}; \frac{x^2}{2} \right)
\]

is \( R_\mu \).

Again, putting \( \nu = -\mu \), we find that

\[
x^{1/2-\mu} \, _1F_1\left( \frac{3}{2}; 3; \frac{x^2}{2} \right)
\]

is \( R_\mu \).

4. Miss Sinha\(^9\) has shown that

\[
\int_0^\infty x^{(l-1)} K_m (ax) J_\nu (bx) J_\delta (bx) \, dx
\]

\[
= \frac{2^{l-2} b^{\nu+\delta} \Gamma \left( \frac{l + \nu + \delta + m}{2} \right) \Gamma \left( \frac{l + \nu + \delta - m}{2} \right)}{a^{l+\nu+\delta} \Gamma (\nu + 1) \Gamma (\delta + 1)}
\]

\[
\times \, _4F_3 \left[ \begin{array}{c}
1 + \nu + \delta, \frac{2 + \nu + \delta}{2}, \frac{l + \nu + \delta + m}{2}, \\
l + \nu + \delta - m, \frac{-4b^2}{a^2}; 1 + \nu, 1 + \delta, 1 + \nu + \delta;
\end{array} \right]
\]

where \( R (l + \nu + \delta + m) > 0, \ R (\delta) > -1, \)

and \( R (\nu) > -1. \)

After adjusting a little, we find that it may be put in the form,

\[
\int_0^\infty (ax)^{l+\nu+\delta-1/2} K_m (ax) J_\nu (bx) J_\delta (bx) \, dx
\]

\[
= \frac{2^{l-2} b^{\nu+\delta} \Gamma \left( \frac{\nu + l + \delta + m}{2} \right) \Gamma \left( \frac{\nu + l + \delta - m}{2} \right)}{\sqrt{a} \, \Gamma (\nu + 1) \Gamma (\delta + 1)}
\]

\[
\times \, _4F_3 \left[ \begin{array}{c}
1 + \nu + \delta, \frac{2 + \nu + \delta}{2}, \frac{l + \nu + \delta + m}{2}, \\
l + \nu + \delta - m, \frac{-4b^2}{a^2}; 1 + \nu, 1 + \delta, 1 + \nu + \delta;
\end{array} \right]
\]

\[(4.2)\]
I have shown in a previous paper\textsuperscript{11} that

\[ J_{\mu}(a_{1}x) J_{\nu}(a_{2}x) \cdots J_{k}(a_{n}x) \]

where \( a_{1} + a_{2} + \cdots + a_{n} \) is finite and \( a_{1}, a_{2}, \ldots, a_{n} \) are all +ve, is \( R_{3} \).

Hence taking \( n = 2 \) and \( a_{1} = a_{2} = b_{1} \) we find that

\[ J_{\mu}(ax) J_{\nu}(bx) \]

is \( R_{1} \) if \( b \) is finite and +ve.

Also Dr. Brij Mohan\textsuperscript{1} has shown that the Kernel

\[ x^{1/2(\mu+p+1)} K_{1/2(p-\mu)}(x), \]

transforms \( R_{\mu} \) into \( R_{p} \).

5. Putting \( \mu = 1 \), we find that the Kernel

\[ x^{1/2(p+2)} K_{1/2(p-1)}(x) \]

transforms \( R_{1} \) into \( R_{p} \).

Hence putting

\[ \frac{1}{2}(p - 1) = m, \]

\[ 1 + \frac{p}{2} = l + \nu + \delta - \frac{1}{2}, \]

we find that

\[ p = m + l + \nu + \delta - 1 = 2m + 1, \]

and

\[ l + \nu + \delta - m = 2. \]

Hence applying the results of (5.6), (5.5) and (5.3) to (5.2), we find that

\[ \frac{1}{\sqrt{x}} \ _{4}F_{3} \left[ \frac{1 + \nu + \delta}{2}, \frac{2 + \nu + \delta}{2}, 1 + m, 1; \right. \left. \frac{-4b^{2}}{x^{\delta}}; 1 + \nu, 1 + \delta, 1 + \nu + \delta; \right], \]

where \( \nu + \delta - m \leq 2 \) is \( R_{2m+1} \).

In other words, we find that

\[ \frac{1}{\sqrt{x}} \ _{4}F_{3} \left[ \frac{1 + \nu + \delta}{2}, \frac{2 + \nu + \delta}{2}, \frac{1 + m}{2}, 1; \right. \left. \frac{-4b^{2}}{x^{\delta}}; 1 + \nu, 1 + \delta, 1 + \nu + \delta; \right], \]
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where \( \nu + \delta - \frac{m}{2} \leq \frac{3}{2} \), is \( R_m \).

As a particular case putting \( \nu + \delta = 0 \), we find that

\[
\frac{1}{\sqrt{x}} \, {}_3F_2 \left[ \frac{1}{2}, \frac{1}{2} + \frac{m}{2}, 1; 1 + \nu, 1 - \nu; \frac{-4b^2}{x^2} \right]
\]

is \( R_m \) if \( b \) is + ve and finite.

Further putting \( \nu = \frac{1}{2} \), we find that

\[
\frac{1}{\sqrt{x}} \, {}_2F_1 \left( \frac{1}{2} + \frac{m}{2}, 1; 1 - \frac{4b^2}{x^2}; \frac{x}{2} \right)
\]

is \( R_m \). In particular, if \( m = 2 \) and \( b = \frac{1}{2} \) it reduces to the \( R_2 \) function

\[
\frac{1}{\sqrt{x}} \, {}_2F_1 \left( 1; 1 - \frac{1}{x^2}; \frac{x^{3/2}}{(x^2 + 1)} \right)
\]

given by me in a previous paper.\(^\text{12}\)

6. Macrobert\(^\text{10}\) has shown that

\[
\sqrt{\frac{\pi}{2}} \Gamma(m + n + 1) \Gamma(m - n) (z^2 - 1)^{-m/2} P_{-m}(z)
\]

\[
= \int_0^\infty e^{-\lambda z} K_{n+1/2}(\lambda) \lambda^{m-1/2} \, d\lambda,
\]

where \( R(z) > -1 \), \( R(m + n + 1) > 0 \),

and \( R(m - n) > 0 \).

Putting \( n = 0 \) and writing \( (m + 1) \) for \( m_1 \), \& \( (z - 1) \) for \( z \) we find that (6.1) may be written as

\[
\Gamma(m + 2) \Gamma(m + 1) (z - 2)^{-m+1/2} P_{-m+1}(z - 1)
\]

\[
= \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-\lambda z} \lambda K_{1/2}(\lambda) (\lambda z)^{m+1/2} \, d\lambda,
\]

where \( R(z) > 0 \) and \( R(m + 1) > 0 \).

Now it is known that

\[
\sqrt{\frac{\pi}{2}} e^z K_{1/2}(z) = \frac{1}{\sqrt{z}}.
\]

Further, Dr. Brij Mohan\(^\text{8}\) has shown that the function

\[
\frac{1}{\sqrt{z}}
\]

(6.4)
is $R_p$. Also, it has been further shown by Dr. Brij Mohan\(^2\) that the Kernel
\[ e^{-z} z^{p-1/2} \]
transforms $R_{p-1}$ into $R_p$.

Applying the results of (6.4) and (6.5) to (6.2), we find that the function
\[ (z - 2)^{-m/2} p_0^{-m} (z - 1) = (z - 2)^{-m/2}, \]
where $p_0^{-m} (z)$ is Associated Legendre Polynomial of the 1st kind is $R_m$.

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**REFERENCES**

12. ——   .. “Self-reciprocal properties of certain functions” (in press).