A FEW SELF-RECIPROCAL FUNCTIONS

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1. THE object of this note is to find out a few Self-Reciprocal Functions for Hankel transforms. I denote a function $f(x)$ as R_μ, if it is Self-Reciprocal for Hankel transforms of order μ so that it is given by

$$f(x) = \int_0^\infty J_\mu (xy) f(y) \sqrt{xy}\, dy, \tag{1.1}$$

where $J_\mu(x)$ is a Bessel function of order μ. If $\mu = \frac{1}{2}$, then $f(x)$ is denoted by R_s, while if it is $R - \frac{1}{2}$, it is written as R_c.

Hardy and Titchmarsh[8] have shown that under certain broad conditions $f(x)$ is R_μ, if

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s/2}\, \Gamma\left(\frac{1}{4} + \frac{\mu}{2} + \frac{s}{2}\right) \psi(s)\, x^{-s}\, ds, \tag{1.2}$$

where $\quad 0 < c < 1,$

and $\quad \psi(s) = \psi(1 - s).$

If $\psi(s) = -\psi(1 - s)$, then $f(x)$ is denoted by $-R_\mu$ and in that case $f(x)$ is said to be Skew reciprocal for Hankel transforms of order μ.

2. We start with the integral given by Watson[12]

$$\int_0^\infty \frac{J_{a+p}(ax)\, J_{a-p-1}(ax)\, dx}{x^\lambda}$$

$$= \frac{a^{\lambda-1}\,\Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right)\Gamma\left(a - \frac{\lambda}{2}\right)}{2\,\Gamma\left(p + \frac{\lambda}{2} + 1\right)\Gamma\left(a + \frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda}{2} - p\right)} \tag{2.1}$$

where $\quad \lambda \neq 0.$

Putting $\lambda = (1-s)$, $d = 3/2$, $p = -\frac{1}{2}$ and $a = 1/\sqrt{2}$, we obtain that

$$\int_0^\infty \left[J_1\left(\frac{x}{\sqrt{2}}\right)\right]^2 t^{s-1}\,dt = \frac{2^{s/2}}{2}\cdot\frac{1}{\Gamma\left(2-\frac{s}{2}\right)}, \tag{2.2}$$

where $s \neq 1$.

$\therefore$ Applying Mellin's Inversion formula[7] we obtain

$$\left[J_1\left(\frac{x}{\sqrt{2}}\right)\right]^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2^{s/2}\,\Gamma\left(\frac{3}{2}+\frac{s}{2}\right)}{2\,\Gamma\left(2-\frac{s}{2}\right)\Gamma\left(\frac{3}{2}+\frac{s}{2}\right)}\,x^{-s}\,ds,$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s/2}\,\Gamma\left(\frac{3}{2}+\frac{s}{2}\right)\psi(s)\,x^{-s}\,ds, \tag{2.3}$$

where $0 < c < 1$,

and $\psi(s) = \psi(1-s)$.

Hence from (2.3) and (1.2) we find that

$$\left[J_1\left(\frac{x}{\sqrt{2}}\right)\right]^2 \tag{2.4}$$

is $R_{5/2}$.

3. Goldstein[6] has shown that

$$\int_0^\infty t^{(l-1)}\,e^{t/2}\,I_\nu(2a\sqrt{t})\,W_{k,m}(t)\,dt$$

$$= \frac{\Gamma\left(\frac{\nu}{2}+l+m+\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}+l-m+\frac{1}{2}\right)}{\Gamma(\nu+1)\,\Gamma\left(\frac{\nu}{2}+l-k+1\right)}$$

$$\times\, a^\nu\,{}_2F_2\left[\begin{matrix}\frac{\nu}{2}+l+m+\frac{1}{2},\ \frac{\nu}{2}+l-m+\frac{1}{2}\,; \\ a^2,\ \nu+1,\ \frac{\nu}{2}l-k+1\,;\end{matrix}\right]$$

where $\left(l+\frac{\nu}{2}-m+\frac{1}{2}\right) > 0$. (3.1)

Putting

$$k=\frac{p}{2}-\frac{5}{4},\quad m=\frac{p}{2}-\frac{1}{4},\quad \frac{t^2}{2} \text{ for } t \quad\text{and}\quad \frac{a}{\sqrt{2}} \text{ for } a,$$

we find that (3·1), becomes

$$\int_0^\infty (at)^{(2l-1)} e^{t^2/4}\, W_{p/2-5/4,\, p/2-1/4}\left(\frac{t^2}{2}\right) I_\nu(at)\, dt$$

$$=\frac{\Gamma\left(\frac{\nu}{2}+l+\frac{p}{2}+\frac{1}{4}\right)\Gamma\left(\frac{\nu}{2}+l-\frac{p}{2}+\frac{3}{4}\right)a^{\nu+2l-1}}{\Gamma(\nu-1)\,\Gamma\left(\frac{\nu}{2}+l-\frac{p}{2}+\frac{9}{4}\right)2^{\nu/2-l+1}}$$

$$\times\ {}_2F_2\left[\begin{matrix}\frac{\nu}{2}+l+\frac{p}{2}+\frac{1}{4},\ \frac{\nu}{2}+l-\frac{p}{2}+\frac{3}{4};\\ \frac{a^2}{2},\ \nu+1,\ \frac{\nu}{2}+l-\frac{p}{2}+\frac{9}{4};\end{matrix}\right]$$

where $\left(l+\frac{\nu}{2}-\frac{p}{2}+\frac{3}{4}\right)>0.$ (3·2)

Bailey[5] has shown that

$$e^{t^2/4}\, W_{p/2-5/4,\, p/2-1/4}\left(\frac{t^2}{2}\right) \tag{3·3}$$

is R_p.

Also Dr. Brij Mohan[4] has shown that the kernel

$$x^{1/2(p+q+1)}\, I_{1/2(p-q)}(x), \tag{3·4}$$

transforms R_p into R_q.

Hence putting

$$\left.\begin{array}{l} \frac{1}{2}(p+q+1)=2l-1, \\ \text{and}\quad \frac{1}{2}(p-q)=\nu, \\ \text{we find that} \\ p=2l+\nu-\frac{3}{2}, \\ \text{and}\quad q=2l-\nu-\frac{3}{2} \end{array}\right\}. \tag{3·5}$$

Applying the results of (3.5), (3.4), and (3.3) to (3.2) we conclude that

$$x^{2\nu+\mu+1/2}\cdot{}_2F_2\left[\begin{matrix}2\nu+\mu+1,\ \frac{3}{2};\ \frac{x^2}{2},\\ \nu+1,\ \ 3;\end{matrix}\right] \tag{3.6}$$

is R_μ.

Some of the particular cases of this function are interesting.

Putting $\nu = 1 - \frac{\mu}{2}$, we find that

$$x^{3/2}\, {}_1F_1\left(2 - \frac{\mu}{2}\,;\ \frac{3}{2}\,;\ \frac{x^2}{2}\right) \tag{3.7}$$

is R_μ,

Again, putting $\nu = -\mu$, we find that

$$x^{1/2-\mu}\, {}_1F_1\left(\frac{3}{2}\,;\ 3;\ \frac{x^2}{2}\right) \tag{3.8}$$

is R_μ.

4. Miss Sinha[9] has shown that

$$\int_0^\infty x^{(l-1)} K_m(ax)\, J_\nu(bx)\, J_\delta(bx)\, dx$$

$$= \frac{2^{l-2}\, b^{\nu+\delta}\, \Gamma\left(\frac{l+\nu+\delta+m}{2}\right) \Gamma\left(\frac{l+\nu+\delta-m}{2}\right)}{a^{l+\nu+\delta}\, \Gamma(\nu+1)\, \Gamma(\delta+1)}$$

$$\times\ {}_4F_3\left[\begin{matrix} \frac{1+\nu+\delta}{2},\ \frac{2+\nu+\delta}{2},\ \frac{l+\nu+\delta+m}{2}, \\ \frac{l+\nu+\delta-m}{2};\ \frac{-4b^2}{a^2};\ 1+\nu,\ 1+\delta,\ 1+\nu+\delta; \end{matrix}\right] \tag{4.1}$$

where $R(l+\nu+\delta \pm m) > 0$, $R(\delta) > -1$,

and $R(\nu) > -1$.

After adjusting a little, we find that it may be put in the form,

$$\int_0^\infty (ax)^{l+\nu+\delta-1/2} K_m(ax)\, \frac{J_\nu(bx)\, J_\delta(bx)}{x^{\nu+\delta+1/2}}\, dx$$

$$= \frac{2^{l-2}\, b^{\nu+\delta}\, \Gamma\left(\frac{\nu+l+\delta+m}{2}\right) \Gamma\left(\frac{\nu+l+\delta-m}{2}\right)}{\sqrt{a}\, \Gamma(\nu+1)\, \Gamma(\delta+1)}$$

$$\times\ {}_4F_3\left[\begin{matrix} \frac{1+\nu+\delta}{2},\ \frac{2+\nu+\delta}{2},\ \frac{l+\nu+\delta+m}{2}, \\ \frac{l+\nu+\delta-m}{2};\ \frac{-4b^2}{a^2};\ 1+\nu,\ 1+\delta,\ 1+\nu+\delta; \end{matrix}\right] \tag{4.2}$$

I have shown in a previous paper[11] that

$$\frac{J_\mu(a_1x)\, J_\nu(a_2x) \ldots J_k(a_nx)}{x^{\mu+\nu+\ldots+k+1/2}},$$

where $a_1 + a_2 + \cdots + a_n$ is finite and $a_1, a_2, \ldots, a_n$ are all + ve, is R_1. Hence taking $n = 2$ and $a_1 = a_2 = b_1$ we find that

$$\frac{J_\mu(ax)\, J_\nu(bx)}{x^{\mu+\nu+1/2}} \tag{4.3}$$

is R_1 if b is finite and + ve.

Also Dr. Brij Mohan[1] has shown that the Kernel

$$x^{1/2(\mu+p+1)} K_{1/2(p-\mu)}(x), \tag{4.4}$$

transforms R_μ into R_p.

5. Putting $\mu = 1$, we find that the Kernel

$$x^{1/2(p+2)} K_{1/2(p-1)}(x) \tag{5.1}$$

transforms R_1 into R_p.

$$\left.\begin{aligned} &\text{Hence putting}\\ &\qquad \tfrac{1}{2}(p-1) = m,\\ &\qquad 1 + \frac{p}{2} = l + \nu + \delta - \frac{1}{2},\\ &\text{we find that}\\ &\qquad p = m + l + \nu + \delta - 1 = 2m + 1,\\ &\text{and}\\ &\qquad l + \nu + \delta - m = 2. \end{aligned}\right\} \tag{5.2}$$

Hence applying the results of (5.6), (5.5) and (5.3) to (5.2), we find that

$$\frac{1}{\sqrt{x}}\, {}_4F_3\left[\begin{matrix} \dfrac{1+\nu+\delta}{2}, \dfrac{2+\nu+\delta}{2}, 1+m, 1; \\ \dfrac{-4b^2}{x^2};\ 1+\nu,\ 1+\delta,\ 1+\nu+\delta; \end{matrix}\right], \tag{5.3}$$

where $\nu + \delta - m \leq 2$ is R_{2m+1}.

In other words, we find that

$$\frac{1}{\sqrt{x}}\, {}_4F_3\left[\begin{matrix} \dfrac{1+\nu+\delta}{2}, \dfrac{2+\nu+\delta}{2}, \dfrac{1}{2}+\dfrac{m}{2}, 1; \\ \dfrac{-4b^2}{x^2};\ 1+\nu,\ 1+\delta,\ 1+\nu+\delta; \end{matrix}\right], \tag{5.4}$$

where $\quad \nu + \delta - \frac{m}{2} \leq \frac{3}{2}$, is R_m.

As a particular case putting $\nu + \delta = 0$, we find that

$$\frac{1}{\sqrt{x}}\, {}_3F_2\left[\frac{1}{2},\ \frac{1}{2} + \frac{m}{2},\ 1;\ 1 + \nu,\ 1 - \nu;\ \frac{-4b^2}{x^2}\right] \tag{5.5}$$

is R_m if b is + ve and finite.

Further putting $\nu = \frac{1}{2}$, we find that

$$\frac{1}{\sqrt{x}}\, {}_2F_1\left(\frac{1}{2} + \frac{m}{2},\ 1;\ \frac{3}{2};\ \frac{-4b^2}{x^2}\right),$$

is R_m. In particular, if $m = 2$ and $b = \frac{1}{2}$ it reduces to the R_2 function

$$\frac{1}{\sqrt{x}}\, {}_1F_0\left(1;\ -\frac{1}{x^2}\right) = \frac{x^{3/2}}{(x^2 + 1)}$$

given by me in a previous paper.[12]

6. Macrobert[10] has shown that

$$\sqrt{\frac{\pi}{2}}\,\Gamma(m + n + 1)\,\Gamma(m - n)\,(z^2 - 1)^{-m/2}\,P_n^{-m}(z)$$

$$= \int_0^\infty e^{-\lambda z}\,K_{n+1/2}(\lambda)\,\lambda^{m-1/2}\,d\lambda, \tag{6.1}$$

where $\quad R(z) > -1,\ R(m + n + 1) > 0,$

and $\quad R(m - n > 0.$

Putting $n = 0$ and writing $(m + 1)$ for m_1, & $(z - 1)$ for z we find that (6.1) may be written as

$$\Gamma(m + 2)\,\Gamma(m + 1)\,(z - 2)^{-(m+1)/2}\,P_0^{-(m+1)}(z - 1)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\lambda z} e^{\lambda}\,K_{1/2}(\lambda)\,(\lambda z)^{m+1/2}\,d\lambda, \tag{6.2}$$

where $\quad R(z) > 0$ and $R(m + 1) > 0.$

Now it is known that

$$\sqrt{\frac{2}{\pi}}\,e^z\,K_{1/2}(z) = \frac{1}{\sqrt{z}}. \tag{6·3}$$

Further, Dr. Brij Mohan[3] has shown that the function

$$\frac{1}{\sqrt{z}} \tag{6.4}$$

is R_ν. Also, it has been further shown by Dr. Brij Mohan[2] that the Kernel

$$e^{-z} z^{\nu-1/2} \tag{6.5}$$

transforms $R_{\nu-1}$ into R_ν.

Applying the results of (6.4) and (6.5) to (6.2), we find that the function

$$(z-2)^{-m/2} p_0^{-m} (z-1) = (z-2)^{-m/2}, \tag{6.6}$$

where $P_0^{-m}(z)$ is Associated Legendre Polynomial of the 1st kind is R_m.

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