SOLUTION OF THE DIRAC EQUATION FOR ELECTROMAGNETIC POTENTIALS

By N. C. Rastogi and Vachaspati

(*Physics Department, University of Lucknow*)

Received January 29, 1959

(Communicated by Dr. P. L. Bhatnagar, F.A.Sc.)

Abstract

Following a work Eliezer (1958), we have solved the Dirac electron equation for electromagnetic potentials. Explicit symmetrical expressions are obtained for the potentials in terms of the electron wave function, its derivatives and a lieanr combination of the potentials.

The Dirac equation,*

$$i\gamma^{\mu}\left(\frac{\partial}{\partial x^{\mu}} + iA_{\mu}\right)\psi + m\psi = 0, \tag{1}$$

is usually solved for the electron wave function, ψ, in terms of the electromagnetic potentials, A^{μ}. Eliezer (1958) has recently suggested that it is also of interest to solve the converse problem, namely, to determine the potentials, A^{μ}, in terms of ψ and its derivatives. He has shown that the four equations that one gets from (1) are not all independent. They satisfy a consistency condition,

$$\psi' J \psi = 0 \tag{2}$$

where prime denotes the transpose matrix and

$$J = \begin{pmatrix} 0 & -\partial_t & -(\partial_x + i\partial_y) & -\partial_z \\ \partial_t & 0 & \partial_z & (\partial_x - i\partial_y) \\ (\partial_x + i\partial_y) & -\partial_z & 0 & \partial_t \\ -\partial_z & -(\partial_x - i\partial_y) & -\partial_t & 0 \end{pmatrix} \tag{3}$$

with

$$(\partial_t, \partial_x, \partial_y, \partial_z) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

* We put $\hbar = 1$, $c = 1$, $e = 1$. Summation in (1) is understood over the suffix μ.

Dirac has given some general arguments to show that J should have the form

$$\mathrm{J} = \beta a_x a_z (\partial_t + a_x \partial_x + a_y \partial_y + a_z \partial_z) \tag{4}$$

where

$$\beta = \gamma^0,\ (a_x, a_y, a_z) = \beta (\gamma^1, \gamma^2, \gamma^3),$$

which is in agreement with (3).

Taking Dirac's representation of the $\vec{\alpha}$ and β matrices, the equation (1) can be written as

$$\begin{aligned}
- \mathrm{A}_0 \psi_1 + \mathrm{A}_x \psi_4 - i\mathrm{A}_y \psi_4 + \mathrm{A}_z \psi_3 + b_1 &= 0 \\
- \mathrm{A}_0 \psi_2 + \mathrm{A}_x \psi_3 + i\mathrm{A}_y \psi_3 - \mathrm{A}_z \psi_4 + b_2 &= 0 \\
- \mathrm{A}_0 \psi_3 + \mathrm{A}_x \psi_2 - i\mathrm{A}_y \psi_2 + \mathrm{A}_z \psi_1 + b_3 &= 0 \\
- \mathrm{A}_0 \psi_4 + \mathrm{A}_x \psi_1 + i\mathrm{A}_y \psi_1 - \mathrm{A}_z \psi_2 + b_4 &= 0
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
b_1 &= (i\partial_t + m)\,\psi_1 + i\partial_z \psi_3 + (i\partial_x + \partial_y)\,\psi_4 \\
b_2 &= (i\partial_t + m)\,\psi_2 + (i\partial_x - \partial_y)\,\psi_3 - i\partial_z \psi_4 \\
b_3 &= i\partial_z \psi_1 + (i\partial_x + \partial_y)\,\psi_2 + (i\partial_t - m)\,\psi_3 \\
b_4 &= (i\partial_x - \partial_y)\,\psi_1 - i\partial_z \psi_2 + (i\partial_t - m)\,\psi_4, \\
(\mathrm{A}^0, \mathrm{A}^1, \mathrm{A}^2, \mathrm{A}^3) &= (\mathrm{A}_0, -\mathrm{A}_1, -\mathrm{A}_2, -\mathrm{A}_3) \\
&= (\mathrm{A}_0, \mathrm{A}_x, \mathrm{A}_y, \mathrm{A}_z),
\end{aligned}$$

and

$$(x^0, x^1, x^2, x^3) = (t, x, y, z).$$

Since only three of these equations are independent, we can solve for any three of the four variables, A_0, A_x, A_y, A_z, in terms of the fourth. Eliezer has given explicit expressions for A_x, A_y, A_z in terms of A_0. This, however, destroys the symmetry among the four potentials. To preserve this symmetry, we introduce a linear combination of these potentials,

$$\begin{aligned}
\mathrm{P} &= c_0 \mathrm{A}_0 + c_1 \mathrm{A}_x + c_2 \mathrm{A}_y + c_3 \mathrm{A}_z \\
&= \sum_{\alpha=1}^{3} c_\alpha \mathrm{A}^\alpha
\end{aligned} \tag{6}$$

and solve for A_0, A_x, A_y, A_z in terms of P. We can get Eliezer's result† by choosing the constants c_a as

$$(c_0,\ c_1,\ c_2,\ c_3) = (1,\ 0,\ 0,\ 0).$$

The calculations are rather long though straightforward. Therefore we give only the results here:—

$$A^{\mu} = \frac{\phi\gamma^{\mu}\,[P - (m - i\gamma^{\mu}\partial_{\mu})\,\Gamma^{\mu} - iD^{\mu}]\,\psi}{\phi S\psi} \tag{7}$$

[$\mu = 0, 1, 2, 3$; no summation over μ here]

where

$$\phi = \psi'\gamma^2\gamma^0 \tag{8 a}$$

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} \tag{8 b}$$

$$\Gamma^{\mu} = \sum_{a\neq\mu} c_a\gamma^a, \qquad a = 0,\ 1,\ 2,\ 3 \tag{8 c}$$

$$D^{\mu} = \sum_{a\neq\mu} c_a \frac{\partial}{\partial x_a}, \qquad x_a = \sum_{\beta=0}^{3} g_{a\beta}x^{\beta} \tag{8 d}$$

$$S = \sum_{a=0}^{3} c_a\gamma^a \tag{8 e}$$

and we use the metric $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$, $g_{\mu\nu} = 0$, $\mu \neq \nu$.

One can get to Eliezer's notation by replacing‡ A_0 by

$-\phi$, $(A_x,\ A_y,\ A_z)$ by $(A_1,\ A_2,\ A_3)$

and

$$\left(\frac{\partial}{\partial x^1},\ \frac{\partial}{\partial x^2},\ \frac{\partial}{\partial x^3}\right)$$

by

$$(-\nabla_x,\ -\nabla_y,\ -\nabla_z).$$

---

† Eliezer's equations (25) and (26) are incorrect. They should, respectively, be

$$R = \alpha_y\beta\,[m\beta + (\phi + i\nabla_t) - \vec{\alpha}\cdot\vec{\nabla}]$$
$$a = \alpha_y\,\beta.$$

Here Eliezer's notation is used. In writing the expression for R, Eliezer uses the fact that $\alpha_y' = -\alpha_y$, $(\beta\alpha_y\alpha_z)' = -\beta\alpha_y\alpha_z$, where primes denote the transpose of the matrix. Consequently $\psi'\alpha_y\psi = 0$ and $\psi'\beta\alpha_y\alpha_z\psi = 0$.

‡ This ϕ is not to be confused with ϕ in (8 a).

ACKNOWLEDGEMENT

One of us (N. C. R.) is grateful to the University of Lucknow for the award of a research training scholarship of the Government of India.

REFERENCE

Eliezer, C. J. .. *Proc. Camb. Phil. Soc.*, 1958, **54**, 247.