REDUCTION OF SOLVABLE GROUPS

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Monomial Representations

Representations of solvable groups can be elegantly studied with the help of monomial representations. Let G be a group of finite order and U be a subgroup of G. Let

$$G = \sum_{r=1}^{k} g_r U$$

be a left coset decomposition of G with respect to U. In this decomposition the choice of the elements g_r is not unique but when they are chosen once, they are taken as fixed for the subsequent discussion. Every element g of G could be written as $g_r u$, where u belongs to U. Consider the product $gg_i = g_r u g_i$. This element could be written as the product of some g_j and u' uniquely, where u' belongs to U. Now construct the matrix whose i-th row j-th column element is u' and all the other elements in the i-th row are zeros. It is easily verified that these matrices define a representation of the group G. More general representations[1] of G could be defined with respect to U and a normal subgroup H of U. These are called the monomial representations of G. We are, however, making use of only representations in which H is always taken as consisting of the identity element only. The monomial representations of G that could be obtained either by taking different sets of elements g_r in the coset decomposition or by taking different subgroups U^1 conjugate to U are all equivalent.

Representations of G when G/U is of Prime Order

Let U be a normal subgroup of G and let G/U be a cyclic group of prime order say r. Then G may be written in the form $G = \sum_{k=0}^{r-1} s^k U$, where s is some element in G not contained in U. Let $R^{(0)}$ be an irreducible representation of U and let the matrix corresponding to an element u of U be $R_u^{(0)}$. From the representation $R^{(0)}$, we obtain the representations $R^{(i)}$ $\{i = 1, 2, \ldots, (r-1)\}$ of U by making u to correspond to the matrix $R_u^{(i)} = R_{s^{-i}us^i}$. If in the monomial representation of G defined with respect to U, the elements of U are replaced by the corresponding matrices in the representation $R^{(0)}$ of U, we obtain a matrix representation R of G and we have the following two theorems given by Seitz.[2]

Theorem 1.—The representations $R^{(i)}$ of U, for different values of i, are all either non-equivalent or equivalent.

Theorem 2.—If all the representations are non-equivalent, R is an irreducible representation of G and if they are all equivalent, R is reducible into r non-equivalent irreducible representations.

Definition.—We define the set of representations $R^{(i)}$ when they are non-equivalent as forming a degenerate star with r arms. When they are equivalent, they may be taken as identical with one of them say $R^{(0)}$ and $R^{(0)}$ itself defines a (non-degenerate) star. A set of irreducible representations of U with one member from each star is defined as a basis set of representations of U with respect to G.

Theorem 3.—The irreducible representations of G which are obtained from a basis set of representations of U, form a complete set of irreducible representations of G.

Proof.—Let U possess a_k, k-dimensional ($k = 1, 2, \ldots, s$) irreducible representations forming a complete set. Then we have the orthogonality relation $\sum_{k=1}^{s} a_k k^2 = N(U)$, the order of the group U. Among these representations of U let b_1 one dimensional representations, b_2 two dimensional representations, etc., be the total number of representations belonging to degenerate stars and let the others belong to the non-degenerate stars. Every degenerate star is of order r. Then from Theorem 2, it follows that G contains $r\,(a_1 - b_1)$ one dimensional, $r\,(a_2 - b_2)$ two dimensional, etc., irreducible representations arising from the non-degenerate star and b_1/r, r-dimensional, b_2/r, $2r$-dimensional, etc., irreducible representations arising from the degenerate stars. All these representations are non-equivalent as they correspond to different stars. Now the relation

$$\sum_{k=1}^{s} r\,(a_k - b_k)\,k^2 + \frac{b_k}{r}\,.\,r^2 = r \sum_{k=1}^{s} a_k \cdot k^2 = r\,N(U),$$

shows that all the representations of G are obtained in the above enumeration.

Solvable Groups

Let G be a solvable group and B a normal Abelian subgroup (preferably largest) of G. The irreducible representations of the Abelian group B are all one dimensional and there are as many non-equivalent irreducible representations of B as there are elements in it. Let $b_p^{(r)}$ be the complex number in the r-th irreducible representation corresponding to the element p of B. Now every element s of G defines a mapping sBs^{-1} of B on to itself. Every such mapping transforms the representation $b^{(r)}$ into a representation in which the element sps^{-1} corresponds to the number $b_p^{(r)}$. The latter representation may be the same as the representation $b^{(r)}$ or may be distinct from it. All the elements of G which transform the representation $b^{(r)}$ into itself form a group $G^{(r)}$ containing B. All the elements of G which transform the representations $b^{(r)}$, $b^{(s)}$, etc., simultaneously into themselves form a group which is the intersection of the groups $G^{(r)}$, $G^{(s)}$, etc. Elements of G belonging to the same coset $s_iG^{(r)}$ transform the representation $b^{(r)}$ into identical representations of B and elements belonging to two different cosets $s_iG^{(r)}$ and $s_jG^{(r)}$ transform $b^{(r)}$ into distinct representations of B. As before, we call the set of all distinct irreducible representations of B which are conjugate with respect to each other in relation to the elements of G, a star and each member an arm of the star. If we select one representation from each star we obtain a basis set of irreducible representations of B defined with respect to G. The groups $G^{(r)}$, $G^{(s)}$, etc., which correspond to different irreducible representations of B in the same star are conjugate subgroups of G.

NON-SPURIOUS REPRESENTATIONS OF $G^{(r)}$

$G^{(r)}$ is a solvable group, as it is a subgroup of the solvable group G. The irreducible representations of $G^{(r)}$ could be obtained from those of B using Theorem 2 by stages. However, it is not necessary to obtain all the irreducible representations of $G^{(r)}$, for complete reduction of G. By the method of Theorem 2, it is sufficient to obtain those representations of $G^{(r)}$ which could be generated from the representation $b^{(r)}$ of B with respect to which the group $G^{(r)}$ itself is defined. These representations are called non-spurious representations. The representations of $G^{(r)}$ which arise from those of B other than $b^{(r)}$ may be called spurious representations. Suppose

$$G^{(r)} = G_0^{(r)} \supset G_1^{(r)} \supset \ldots \supset B$$

be a composition series of $G^{(r)}$ containing the group B. Using the argument of Theorem 3 the following result may easily be proved.

Theorem 4.—If there are a_k, k-dimensional non-spurious representations of $G_t^{(r)}$ that could be obtained from the representation $b^{(r)}$ of B

$$\Sigma\, a_k \cdot k^2 = \frac{N\,(G_t^{(r)})}{N\,(B)}$$

where the summation extends over all the non-spurious representations of $G_t^{(r)}$.

It may be noted that in all these non-spurious representations of $G_t^{(r)}$, for every t in the composition series, the matrix corresponding to an element p of B is a scalar matrix with the scalar multiple $b_p^{(r)}$.

Suppose $b^{(s)}$ and $b^{(t)}$ be two different irreducible representations of B, then we have the following theorem.

Theorem 5.—The irreducible representations of $A = G^{(s)} \cap G^{(t)}$ induced by the non-spurious representations of $G^{(s)}$ are distinct from those induced by $G^{(t)}$ when $s \neq t$.

Proof.—The Abelian group B is a subgroup of A and in the representations induced by the non-spurious representations of $G^{(s)}$ and $G^{(t)}$, scalar matrices with different scalar multiples $b_p^{(s)}$ and $b_p^{(t)}$ correspond to an element p of B respectively. Thus the representations of A are non-equivalent. Let $M^{(s)}$ and $M^{(t)}$ be two non-spurious representations of $G^{(s)}$ and $G^{(t)}$ respectively. From the above theorem we have

$$\Sigma\ (tr M_a^{(s)})\,(tr^*\, M_a^{(t)}) = 0 \tag{7}$$

the summation extending over all elements of A.

Representations of G

Let a left coset decomposition of G with respect to $G^{(r)}$ be given by

$$G = \sum_{i=1}^{r} s_i G^{(r)}.$$

In the monomial representation of G using the above coset decomposition, the matrix corresponding to $s_t g^{(r)}$ contains in the i-th row, i-th column (*i.e.*, the i-th diagonal element) an element $g'^{(r)}$ provided that

$$s_t g^{(r)} s_i = s_i g'^{(r)}.$$

Then $s_t g^{(r)}$ belongs to the group $s_i G^{(r)} s_i^{-1}$, which is conjugate to $G^{(r)}$. Conversely, every element g of G belonging to the group $s_i G^{(r)} s_i^{-1}$, is of the form $s_i g^{(r)} s_i^{-1}$, and will have in the monomial representation of G with respect to $G^{(r)}$, the element $g^{(r)}$ as the i-th diagonal element. In the monomial representation the matrix corresponding to g will have as many diagonal elements as there are subgroups $s_t G^{(r)} s_t^{-1}$ which contain the element g.

Theorem 6.—The matrix representation R of G, obtained from a monomial representation of G with respect to $G^{(r)}$ making use of a non-spurious representation of $G^{(r)}$, is an irreducible representation of G.

Proof.—Let $M^{(\tau_1)}$ be a non-spurious representation of $G^{(\tau_1)} = G^{(\tau)}$ and $M^{(\tau_i)}$ that of $s_i G^{(\tau)} s_i^{-1} = G^{(\tau_i)}$ which is obtained from the non-spurious representation of $G^{(\tau_1)}$ by the transformation of $M^{(\tau_1)}$ by s_i. Let $tr\, M_g^{(\tau_i)}$ be the trace of the element g in the representation $M^{(\tau_i)}$ if g belongs to the group $G^{(\tau_i)}$ and zero otherwise. Then

$$tr R_g = \sum_{i=1}^{k} tr M_g^{(\tau_i)}. \tag{8}$$

Now,

$$\begin{aligned}\sum_g (tr R_g)(tr^* R_g) &= \sum_g \Big(\sum_{i=1}^{k} tr M_g^{(\tau_i)}\Big)\Big(\sum_{i=1}^{k} tr^* M_g^{(\tau_j)}\Big) \text{ from (8)}\\ &= \sum_g \sum_{i,j} (tr M_g^{(\tau_i)})(tr^* M_g^{(\tau_j)})\\ &= \sum_{i,j} \sum_g (tr M_g^{(\tau_i)})(tr^* M_g^{(\tau_j)})\\ &= \sum_i N(G^{(\tau_i)}) \quad \text{by Theorems 4 and 5}\\ &= N(G).\end{aligned}$$

This shows that the representation R of G is irreducible.

Theorem 7.—Two representations of G which are obtained from two different non-spurious representations $M^{(\tau)}$ and $M'^{(\tau)}$ of $G^{(\tau)}$ are non-equivalent.

Proof.—As in the previous theorem this follows from the fact

$$\sum_g (tr R_g)(tr^* R_g') = 0.$$

Similarly, by making use of Theorem 9 we can prove that two representations of G which are obtained from the non-spurious representations of the groups $G^{(\tau)}$ and $G^{(s)}$ corresponding to two different representations $b^{(\tau)}$ and $b^{(s)}$ in a basis set of representations of U with respect to G, are orthogonal. Finally, we arrive at the following theorem.

Theorem 8.—The representations of G, which are obtained from the non-spurious representations of all the subgroups $G^{(\tau)}$ corresponding to all representations $b^{(\tau)}$ in a basis set of representations of B defined with respect to G, form a complete set.

Proof.—B is an Abelian group and all its irreducible representations are one dimensional. They have been classified into different stars using the elements of G. A basis set of representations $b^{(\tau)}$, $b^{(s)}$, etc., is obtained by taking one representation from each star. If the degeneracy of the star containing $b^{(r)}$ is d_r, we have

$$\sum_r d_r = N(B). \tag{8 a}$$

The number of cosets of $G^{(r)}$ in G is equal to d_r and we have also the relation

$$d_r \times N(G^{(r)}) = N(G). \tag{9}$$

The dimension of a matrix representation of G, obtained from the monomial representation of G with respect to $G^{(r)}$, using for the elements of $G^{(r)}$ their corresponding matrices in a non-spurious representation of dimension n will be nd_r. Hence, if there are $a_s^{(r)}$, s-dimensional non-spurious representations of $G^{(r)}$, they give rise to sd_r-dimensional representations of G ($s = 1, 2, 3$, etc.). Therefore we have

$$\sum_s a_s^{(r)} (sd_r)^2 = \frac{d_r^2 N(G^{(r)})}{N(B)} \qquad \text{by Theorem 4.}$$

Now summing over r we have

$$\frac{\sum_r d_r^2 N(G^{(r)})}{N(B)} = \frac{\sum_r d_r \{d_r N(G^{(r)})\}}{N(B)}$$

$$= \frac{N(G)\left(\sum_r d_r\right)}{N(B)} \qquad \text{from (9)}$$

$$= N(G). \qquad \text{from (8)}$$

Hence the theorem.

Summary

In this paper a method of obtaining the representations of solvable groups, making use of the non-spurious representations defined with respect to the representations of a normal Abelian subgroup is given and the completeness of these representations is established.

References

1. Hans Zassenhaus .. *Theory of Groups*, 1949, 136.
2. Seitz, F. .. *Annals of Mathematics*, 1937, **37**, 17.