THE DIFFRACTION OF LIGHT BY SUPERPOSED PARALLEL SUPERSONIC WAVES GENERAL THEORY

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1. INTRODUCTION

THE first theoretical investigations of the diffraction of light by superposed supersonic waves in liquids and solids date from 1938. Theories were then developed by Bergmann and Fues,[1] Nagendra Nath[2] and Nagabhushana Rao[3] and were mainly based on Raman and Nath's preliminary theory.[4] The considered sound waves have frequencies which are incommensurate and they do not present phase differences, so that the expressions for the intensities of the spectra derived by those authors only have a limited use.

Experiments showing combination spectra were performed by Bergmann[5] (in liquids), Hiedemann and Hoesch[6] (in solids), Bergmann and Fues[1] (in liquids and solids) and Govinda Rao[7] (in liquids). None of those experimental studies gives quantitative measurements of the intensities of the obtained spectra.

In 1949 Ramachandra Rao[8] set up a theory of the diffraction of light by two superposed parallel ultrasonic waves, the wavelength of the second being half the wavelength of the first (or the ratio of the frequency of the first to the frequency of the second being 1 : 2; there also exists a definite phase relationship between the two sound waves. Using the method of Raman-Nath's simplified theory and confining himself to a perturbation showing no time dependence, he comes to the interesting conclusion that the obtained spectra must be asymmetric with respect to the central image. This feature has been confirmed experimentally by the same author.

Those calculations were extended by Murty[9] to the case of two sound waves, the frequencies of which are generally in the ratio 1 : *n*. This theory which is still based on Raman-Nath's elementary theory leads to the following conclusions:

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When the ratio of the frequencies is $1:(2l+1)$ the intensities of opposite orders are equal so that the pattern is symmetric;

When the ratio of the frequencies is $1:2l$ the intensities of opposite orders are different and the pattern is asymmetric.

Recently the interest in the problem of the diffraction of light by superposed parallel supersonic waves was again revived by the study of the distortion of finite amplitude sound waves by Zankel and Hiedemann[10]: the distorted wave is resolved into its Fourier components, so that the latter can be considered as superposed waves.

The purpose of the present paper is to expose the general theory of the diffraction of light by two superposed parallel supersonic waves, one being the n-th harmonic of the other. In the second section a system of difference–differential equations for the amplitudes of the different spectra is derived. In §3 the special case $\rho \equiv \lambda^2/\mu_0\mu_1\lambda^{*\,2} = 0$ is considered and the system of difference–differential equations is solved exactly. In the following section it is shown that in the case $\rho \neq 0$ the diffraction pattern is always asymmetric with respect to the zero order, as well for even as for odd values of n. Finally (§§ 5–7) a series solution of the general system is given for $n = 2$ and $n = 3$. In the latter case coefficients are calculated up to the term giving the asymmetry effect in the intensity.

It is a great pleasure for us to pay tribute to Prof. Sir C. V. Raman for the fundamental theories for the diffraction of light by supersonic waves he has established and which are characterized by the simplicity of the basic ideas underlying them. Both his preliminary and generalized theories were the first to give a satisfactory explanation of the diffraction patterns containing a great number of spectra and formed the basis for many theoretical and experimental studies, the present investigation included.

2. Establishment of a System of Difference–Differential Equations for the Amplitudes of the Diffracted Light Waves*

Let us consider a parallel beam of light of frequency ν and wavelength in vacuum λ, passing through a liquid column, disturbed by two superposed parallel progressive supersonic waves, one of them being the n-th harmonic of the other. We further suppose that the directions of propagation of the

* Independently and using the same method, Zankel[11] has obtained a system of difference-differential equations in the case where a complete set of harmonics, without phase differences, are contained in the sound wave. For this system, which is a generalization of our equation (9), he has found the exact solution for $\rho = 0$. If only one harmonic, containing a phase constant, is present Zankel has also obtained independently the exact solution (17) of Section 3, in the case $\rho = 0$.

light and sound waves are perpendicular to each other. We shall put the z-axis along the direction of the incident light and the x-axis along the direction of the sound waves.

In order to obtain the differential system for the amplitudes of the different spectra we shall follow the method of Raman and Nath's generalized theory.[12]

Writing for the electric field of the light

$$E(x, z, t) = e^{2\pi i \nu t} \phi(x, z, t), \tag{1}$$

$\phi(x, z, t)$ satisfies the equation

$$\Delta \phi = -\left(\frac{2\pi}{\lambda}\right)^2 \mu^2 (x, t) \phi; \tag{2}$$

$\mu(x, t)$ is the refractive index of the disturbed liquid and is given by the expression

$$\mu(x, t) = \mu_0 + \mu_1 \sin 2\pi \left(\nu^* t - \frac{x}{\lambda^*}\right)$$
$$+ \mu_n \sin 2\pi \left(n\nu^* t - \frac{nx}{\lambda^*} + \Delta\right) \tag{3}$$

where

λ^* = wavelength of the supersonic waves in the considered medium;

ν^* = frequency of the supersonic waves;

μ_0 = refractive index of the undisturbed medium;

μ_1 = maximum variation of the refractive index of the disturbed medium due to the fundamental tone of the supersonic wave;

μ_n = maximum variation of the refractive index of the disturbed medium caused by the n-th harmonic of the supersonic wave;

Δ = phase of the second wave with respect to the first.

According to the periodicity of $\phi(x, z, t)$ in x and t, this function may be developed in a double Fourier series

$$\phi(x, z, t) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} f_{rs}(z)\, e^{2\pi i r\, x/\lambda^*} e^{2\pi i s \nu^* t} \tag{4}$$

Taking into account that $\phi(x + p\lambda^*, z, t) = \phi(x, z, t - p/\nu^*)$, p being any number, it follows that $f_{rs}(z) \equiv 0$ when $r \neq -s$ and $f_{rs}(z) \neq 0$ for $r = -s$.

Hence,

$$\phi(x, z, t) = \sum_{r=-\infty}^{\infty} f_r(z)\, e^{2\pi i r\, x/\lambda^*} e^{-2\pi i r \nu^* t}. \tag{5}$$

Substituting this series into the differential equation (2), equating the coefficients of equal exponentials on both sides, putting:

$$f_r = e^{-2\pi i\, \mu_0 z/\lambda} \phi_r(z) \tag{6}$$

into the result, and neglecting the second derivative of $\phi_r(z)$ with respect to the other terms, we finally obtain the following system of difference-differential equations,

$$2\frac{d\phi_r}{dz} - \sigma_1(\phi_{r-1} - \phi_{r+1}) - \sigma_n(\phi_{r-n}e^{-i\delta} - \phi_{r+n}e^{i\delta}) = i\beta r^2 \phi_r \tag{7}$$

with

$$\sigma_1 = \frac{2\pi\mu_1}{\lambda},\ \sigma_n = \frac{2\pi\mu_n}{\lambda},\ \beta = \frac{2\pi\lambda}{\mu_0\lambda^{*2}},\ \delta = 2\pi\Delta \tag{7 a}$$

or putting

$$\xi = \sigma_1 z = \frac{2\pi\mu_1 z}{\lambda},\ \rho = \frac{\beta}{\sigma_1} = \frac{\lambda^2}{\mu_0\mu_1\lambda^{*2}},\ a_n = \frac{\sigma_n}{\sigma_1} = \frac{\mu_n}{\mu_1} \tag{8}$$

$$\boxed{2\frac{d\phi_r}{d\xi} - (\phi_{r-1} - \phi_{r+1}) - a_n(\phi_{r-n}e^{-i\delta} - \phi_{r+n}e^{i\delta}) = i\rho r^2 \phi_r.} \tag{9}$$

From (5) and (6) it is seen that the r-th order spectrum makes an angle θ_r, defined by $\sin\theta_r = -r(\lambda/\lambda^*)$, with respect to the z-axis, and undergoes a frequency change $-r\nu^*$; its amplitude is proportional with ϕ_r, the factor of proportionality having modulus one.

The boundary conditions are

$$\phi_r(0) = \delta_{r0} \tag{10}$$

where δ_{r0} is the well-known Kronecker symbol.

3. Special Case: $\rho = 0$

For great values of μ^1 and λ^* the right-hand side of equation (9) is negligible and the system becomes

$$2\frac{d\phi_r}{d\xi} - (\phi_{r-1} - \phi_{r+1}) - a_n(\phi_{r-n}e^{-i\delta} - \phi_{r+n}e^{i\delta}) = 0. \tag{11}$$

In order to integrate the system (11) with the boundary conditions (10) we shall use the complex function method of Nagendra Nath-Wilson.[2] We shall put

$$\phi_r(\xi) = \frac{1}{2\pi i} \oint F(\eta) \frac{e^{\xi f(\eta)}}{\eta^{r+1}} d\eta \tag{12}$$

where $F(\eta)$ and $f(\eta)$ are yet unknown functions of the complex variable η, and the integral is taken along a closed path in the complex η-plane, encircling the origin once in the positive sense. According to the boundary conditions (10) it is required that the function $F(\eta)$ must be

$$F(\eta) = 1. \tag{13}$$

Substitution of the expression (12) into the equation (11) gives

$$f(\eta) = \frac{1}{2}\left(\eta - \frac{1}{\eta}\right) + \frac{a_n}{2}\left(\eta^n e^{-i\delta} - \frac{1}{\eta^n e^{-i\delta}}\right) \tag{14}$$

so that the solution becomes

$$\phi_r(\xi) = \frac{1}{2\pi i} \oint \frac{\exp.\left[\frac{\xi}{2}\left(\eta - \frac{1}{\eta}\right)\right] \exp.\left[\frac{a_n\xi}{2}\left(\eta^n e^{-i\delta} - \frac{1}{\eta^n e^{-i\delta}}\right)\right]}{\eta^{r+1}} \times d\eta. \tag{15}$$

Using the well-known Laurent series for the exponential functions[13]

$$\exp.\left[\frac{\xi}{2}\left(\eta - \frac{1}{\eta}\right)\right] = \sum_{p=-\infty}^{\infty} J_p(\xi)\, \eta^p \tag{16 a}$$

$$\exp.\left[\frac{a_n\xi}{2}\left(\eta^n e^{-i\delta} - \frac{1}{\eta^n e^{-i\delta}}\right)\right] = \sum_{q=-\infty}^{\infty} J_q(a_n\xi)\, \eta^{nq} e^{-iq\delta}. \tag{16 b}$$

[$J_n(z)$ being the Bessel function of order n]
in the right-hand side of (15) we obtain

$$\phi_r(\xi) = \frac{1}{2\pi i} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} J_p(\xi)\, J_q(a_n\xi)\, e^{-iq\delta} \oint \eta^{p+rq-r-1}\, d\eta.$$

The integral being equal to $2\pi i \delta_{p,\, r-nq}$, we finally have

$$\boxed{\phi_r(\xi) = \sum_{q=-\infty}^{\infty} J_{r-nq}(\xi)\, J_q(a_n\xi)\, e^{-iq\delta}} \tag{17}$$

which are in fact the solutions obtained by Murty, using the method of Raman and Nath's preliminary theory, but written in a more condensed form. Calculating the intensities, Murty finds that in the case of odd values of n, the intensities of the spectra of orders $+ r$ and $- r$ are equal, so that the spectra are symmetric with respect to the spectrum of order zero; in the case of even values of n the spectra are asymmetric with respect to the order zero, *i.e.*, the intensities of the orders $+ r$ and $- r$ are not equal (excepted for $\delta = \pi/2$).

4. Properties of the Intensities in the General Case ($\rho \neq 0$)

We shall now show that Murty's conclusion for odd values of n will not hold in the general case.

Let us suppose that the intensities of the orders r and $- r$ should be equal, *i.e.*, $I_r = I_{-r}$, then we should have either $\phi_r = e^{i\gamma_r(\xi)} \phi_{-r}$ or $\phi_r = e^{i\gamma_r(\xi)} \phi_{-r}{}^*$.

(*a*) *First possibility*:

$$\phi_r = e^{i\gamma_r(\xi)} \phi_{-r}.$$

Introducing this substitution into the general equation (9) we obtain

$$2 \frac{d\phi_{-r}}{d\xi} - [\phi_{-r+1} e^{i(\gamma_{r-1}-\gamma_r)} - \phi_{-r-1} e^{i(\gamma_{r+1}-\gamma_r)}]$$
$$- a_n [\phi_{-r+n} e^{i(\gamma_{r-n}-\gamma_r)} e^{-i\delta} - \phi_{-r-n} e^{i(\gamma_{r+n}-\gamma_r)} e^{i\delta}]$$
$$= \left(- 2i \frac{d\gamma_r}{d\xi} + i\rho r^2\right) \phi_{-r}. \tag{18}$$

On the other hand, changing r into $- r$ in equation (9) gives

$$2 \frac{d\phi_{-r}}{d\xi} - (\phi_{-r-1} - \phi_{-r+1}) - a_n (\phi_{-r-n} e^{-i\delta} - \phi_{-r+n} e^{i\delta})$$
$$= i\rho r^2 \phi_{-r}. \tag{19}$$

Equations (18) and (19) must be identic, hence,

(1°) $d\gamma_r/d\xi = 0$ from which it follows that γ_r is a constant;

(2°) $e^{i(\gamma_{r-1}-\gamma_r)} = - 1$ and $e^{i(\gamma_{r+1}-\gamma_r)} = - 1$, from which

$$\gamma_r = r\pi; \tag{20}$$

(3°) $e^{i(\gamma_{r-n}-\gamma_r)} e^{-i\delta} = - e^{i\delta}$ and $e^{i(\gamma_{r+1}-\gamma_r)} e^{i\delta} = - e^{-i\delta}$,

from which taking into account (20),

$$\cos \delta = 0 \text{ or } \delta = (k + \tfrac{1}{2}) \pi \text{ if } n \text{ is even,}$$

and

$$\sin \delta = 0 \text{ or } \delta = k\pi \text{ when } n \text{ is odd.}$$

(*b*) *Second possibility*:

$$\phi_r = e^{i\gamma_r(\xi)}\phi_{-r}^*.$$

Making this substitution into the system (9) gives

$$2\frac{d\phi_{-r}^*}{d\xi} - [\phi^*_{-r+1}e^{i(\gamma_{r-1}-\gamma_r)} - \phi^*_{-r-1}e^{i(\gamma_{r+1}-\gamma_r)}]$$

$$- a_n[\phi^*_{-r+n}e^{-i\delta}e^{i(\gamma_{r-n}-\gamma_r)} - \phi^*_{-r-n}e^{i\delta}e^{i(\gamma_{r+n}-\gamma_r)}]$$

$$= i\left(\rho r^2 - 2\frac{d\gamma_r}{d\xi}\right)\phi^*_{-r}. \tag{21}$$

Changing r into $-r$ in the equation (9) and taking its complex conjugate, leads to

$$2\frac{d\phi_{-r}^*}{d\xi} - (\phi^*_{-r-1} - \phi^*_{-r+1}) - a_n(\phi^*_{-r-n}e^{i\delta} - \phi^*_{-r+n}e^{-i\delta})$$

$$= -i\rho r^2\phi_{-r}^*. \tag{22}$$

The identity of the second members of the equations (21) and (22) gives $d\gamma_r/d\xi = r^2\rho$ frcm which $\gamma_r = \rho r^2\xi + r\pi$ (when $\rho = 0$, then $\gamma_r = r\pi$). The first bracket in the left-hand side of equation (21) then reads $\phi^*_{-r-1}e^{i\rho\xi(1+2r)} - \phi^*_{-r+1}e^{i\rho\xi(1-2r)}$. Comparing this expression with the first bracket of the left-hand side of equation (12), it seems that their identity could never be realized. Hence, we may conclude *in the general case*, $\rho \neq 0$: *if* n *is even and* $\delta \neq (k + \frac{1}{2})\pi$ *or if* n *is odd and* $\delta \neq k\pi$ *the intensities of the orders* r *and* $-$ r *could never be equal.*

In the case $\rho = 0$, the same reasoning leads to the conclusion that the intensities of orders r and $-r$ cannot be equal if n is even. When n is odd, the second possibility predicts that the intensities of orders r and $-r$ could be equal and that the relation between ϕ_r and ϕ_{-r} should be $\phi_r = e^{i\pi r}\phi^*_{-r} = (-1)^r\phi_{-r}^*$. It is easy to verify that the solution (17) obeys this relation for odd values of n.

Ramachandra Rao's experimental observation[8] of a symmetric spectrum for $n = 3$ could be explained by the fact that his experimental conditions are so that $\rho \ll 1$ or that the phase angle $\delta = k\pi$.

5. Some General Considerations on the Series Solution of the General System (9)

Introducing the series $\phi_r = \sum_{k=0}^{\infty} A_{rk}\xi^k$ into the system (9), leads after careful inspection to the conclusion that the series must have the following form:

(*a*) $n = 2l + 1$, $m = 0, 1, 2, \ldots\ldots$

$$\left.\begin{aligned}
\phi_0 &= \sum_{k=0}^{\infty} A_{0k}\xi^k \\
\phi_{\pm(2l+1)m\pm1} &= \xi^{m+1}\sum_{k=0}^{\infty} A_{\pm(2l+1)m\pm1,k}\xi^k \\
\phi_{\pm(2l+1)m\pm2} &= \xi^{m+2}\sum_{k=0}^{\infty} A_{\pm(2l+1)m\pm2,k}\xi^k \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\phi_{\pm(2l+1)m\pm l} &= \xi^{m+l}\sum_{k=0}^{\infty} A_{\pm(2l+1)m\pm l,k}\xi^k \\
\phi_{\pm(2l+1)m\pm(l+1)} &= \xi^{m+l+1}\sum_{k=0}^{\infty} A_{\pm(2l+1)m\pm(l+1),k}\xi^k \\
\phi_{\pm(2l+1)m\pm(l+2)} &= \xi^{m+l}\sum_{k=0}^{\infty} A_{\pm(2l+1)m\pm(l+2),k}\xi^k \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\phi_{\pm(2l+1)m\pm(2l+1)} &= \xi^{m+1}\sum_{k=0}^{\infty} A_{\pm(2l+1)m\pm(2l+1),k}\xi^k
\end{aligned}\right\} \qquad (23)$$

(*b*) $n = 2l$, $m = 0, 1, 2, \ldots\ldots$

$$\left.\begin{aligned}
\phi_0 &= \sum_{k=0}^{\infty} A_{0k}\xi^k \\
\phi_{\pm2lm\pm1} &= \xi^{m+1}\sum_{k=0}^{\infty} A_{\pm2lm\pm1,k}\xi^k \\
\phi_{\pm2lm\pm2} &= \xi^{m+2}\sum_{k=0}^{\infty} A_{\pm2lm\pm2,k}\xi^k \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\phi_{\pm2lm\pm l} &= \xi^{m+l}\sum_{k=0}^{\infty} A_{\pm2lm\pm l,}\xi^k \\
\phi_{\pm2lm\pm(l+1)} &= \xi^{m+l}\sum_{k=0}^{\infty} A_{\pm2lm\pm(l+1),k}\xi^k \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\phi_{\pm2lm\pm2l} &= \xi^{m+1}\sum_{k=0}^{\infty} A_{\pm2lm\pm2l,k}\xi^k.
\end{aligned}\right\} \qquad (24)$$

It may easily be verified that the lowest powers of ξ in those series are the same as those in the solution (17) of the system in the special case $\rho = 0$.

6. Explicit Calculation of the Intensities by the Series Method in the Case $n = 2$

According to the series expansions (24) we may write for $n = 2$,

$$\left.\begin{aligned} \phi_0 &= \sum_{k=0}^{\infty} A_{0k}\xi^k \\ \phi_{\pm 2m \pm 1} &= \frac{a_2{}^m \xi^{m+1}}{2^{m+1} m!} \sum_{k=0}^{\infty} A_{\pm 2m \pm 1, k}\xi^k \\ \phi_{\pm 2m \pm 2} &= \frac{a_2{}^{m+1} \xi^{m+1}}{2^{m+1}(m+1)!} \sum_{k=0}^{\infty} A_{\pm 2m \pm 2, k}\xi^k. \end{aligned}\right\} \qquad (25)$$

The factors $a_2{}^m/2^{m+1}m!$ and $a_2{}^{m+1}/2^{m+1}(m+1)!$ are written for convenience and are chosen as the corresponding lowest power coefficients of the series expansion of the solution (17) for $\rho = 0$. Substituting the series (25) into the equation (9) and equating the coefficients of equal powers of ξ on both sides, we find the following relations between the coefficients A_{rk}:

$$\begin{aligned} &(m+k+2) A_{\pm 2m \pm 1, k+1} \mp A_{\pm 2m, k+1} \pm \frac{a_2}{2(m+1)} A_{\pm 2m \pm 2, k} \\ &\qquad \mp m A_{\pm 2m \mp 1, k+1} e^{\mp i\delta} \pm \frac{a_2{}^2}{4(m+1)} A_{\pm 2m \pm 3, k-1} e^{\pm i\delta} \\ &= \frac{i\rho}{2}(2m+1)^2 A_{\pm 2m \pm 1, k} \qquad (m \geqslant 1) \end{aligned} \qquad (26)$$

$$\begin{aligned} &(k+2) A_{\pm 1, k+1} \mp A_{0, k+1} \pm \frac{a_2}{2} A_{\pm 2, k} \mp \frac{a_2}{2} A_{\mp 1, k} e^{\mp i\delta} \\ &\qquad \pm \frac{a_2{}^2}{4} A_{\pm 3, k-1} e^{\pm i\delta} \\ &= \frac{i\rho}{2} A_{\pm 1, k} \end{aligned} \qquad (27)$$

$$\begin{aligned} &(m+k+2) A_{\pm 2m \pm 2, k+1} \mp \frac{m+1}{2a_2} A_{\pm 2m \pm 1, k} \pm \tfrac{1}{4} A_{\pm 2m \pm 3, k-1} \\ &\qquad \mp (m+1) A_{\pm 2m, k+1} e^{\mp i\delta} \pm \frac{a_2{}^2}{4(m+2)} A_{\pm 2m \pm 4, k-1} e^{\pm i\delta} \\ &= \frac{i\rho}{2}(2m+2)^2 A_{\pm 2m \pm 2, k} \qquad (m \geqslant 0) \end{aligned} \qquad (28)$$

$$(k+1)A_{0,k+1} - \tfrac{1}{4}A_{-1,k-1} + \tfrac{1}{4}A_{1,k-1} - \frac{a_2^2}{4}A_{-2,k-1}e^{-i\delta}$$

$$+ \frac{a_2^2}{4}A_{2,k-1}e^{i\delta} = 0. \tag{29}$$

It is seen that it is possible to go over from the relations with the upper signs to the relations with the lower signs by means of the substitution

$$\left.\begin{array}{lll} A_{rk} & \rightarrow & (-1)^r A_{-r,k} \\ \delta & \rightarrow & -\delta + \pi. \end{array}\right\} \tag{30}$$

Hence, if one has obtained the formula for a A_{rk}, the same transformation allows the immediate calculation of $A_{-r,k}$.

From the relations (26)-(29) we have calculated the general expressions for the first three coefficients. We refer to the appendix for an example of the detailed computation of one of the coefficients.

$$A_{2m+2,0} = e^{-i(m+1)\delta} \tag{31 a}$$

$$A_{2m+1,0} = e^{-im\delta} \tag{31 b}$$

$$A_{2m+2,1} = \frac{m+1}{4a_2}e^{-im\delta} + \frac{i\rho}{3}(m+1)(2m+3)e^{-i(m+1)\delta} \tag{31 c}$$

$$A_{2m+1,1} = -\frac{a_2}{2(m+1)}e^{-i(m+1)\delta} + \frac{i\rho}{12}(8m^2+12m+3)e^{-im\delta}$$

$$+ \frac{m}{12a_2}e^{-i(m-1)\delta} \tag{31 d}$$

$$A_{2m+2,2} = -\tfrac{1}{4}e^{-i(m+1)\delta} - \frac{a_2^2}{4(m+2)}e^{-i(m+1)\delta}$$

$$+ \frac{m(m+1)}{96a_2^2}e^{-i(m-1)\delta}$$

$$+ \frac{i\rho}{24a_2}(m+1)(4m^2+10m+5)e^{-im\delta}$$

$$- \frac{\rho^2}{90}(m+1)(2m+3)(2m+5)(5m+4)e^{-i(m+1)\delta}$$

$$\tag{31 e}$$

$$A_{2m+1,2} = -\tfrac{1}{8}\, e^{-im\delta} - \frac{\alpha_2^2}{4(m+1)}\, e^{-im\delta} + \frac{m(m-1)}{480\alpha_2^2}\, e^{-i(m-2)\delta}$$

$$- \frac{i\rho\alpha_2}{24}\,\frac{8m^2+12m+7}{m+1}\, e^{-i(m+1)\delta}$$

$$+ \frac{i\rho}{144\alpha_2}\, m\,(8m^2+12m+1)\, e^{-i(m-1)\delta}$$

$$- \frac{\rho^2}{360}\,(80m^4+304m^3+328m^2+113m+15)\; e^{-im\delta} \tag{31f}$$

From those formulæ we may obtain the expressions for the intensities of the spectra, as far as the third term included of the series expansion. The final formulæ are:

$$I_{\pm 2m\pm 1} = \frac{\xi^{2m+2}\alpha_2^{2m}}{2^{2m+2}(m!)^2}\left\{1 \pm \xi\left(-\frac{\alpha_2}{m+1} + \frac{m}{6\alpha_2}\right)\cos\delta\right.$$

$$+ \xi^2\left[-\frac{1}{4} - \frac{\alpha_2^2}{4}\,\frac{2m+1}{(m+1)^2} + \frac{m^2}{144\alpha_2^2}\right.$$

$$- \frac{m}{12(m+1)}\cos 2\delta + \frac{m(m-1)}{240\,\alpha_2^2}\cos 2\delta$$

$$+ \frac{\rho}{3}\left(-\frac{\alpha_2}{m+1} + \frac{m}{12\alpha_2}\right)\sin\delta$$

$$\left.\left. - \frac{\rho^2}{720}(256m^3+352m^2+92m+15)\right] + \ldots\ldots\right\}. \tag{32}$$

$$I_{\pm 2m\pm 2} = \frac{\xi^{2m+2}\alpha_2^{2m+2}}{2^{2m+2}[(m+1)!]^2}\left\{1 \pm \xi\,\frac{m+1}{2\alpha_2}\cos\delta\right.$$

$$+ \xi^2\left[-\frac{1}{2} - \frac{\alpha_2^2}{2(m+2)} + \frac{(m+1)^2}{16\alpha_2^2}\right.$$

$$+ \frac{m(m+1)}{48\alpha_2^2}\cos 2\delta + \frac{\rho}{12\alpha_2}(m+1)\sin\delta$$

$$\left.\left. - \frac{\rho^2}{45}(m+1)(2m+3)(8m+5)\right] + \ldots\ldots\right\}. \tag{33}$$

Those series show an asymmetry in the spectra with regard to the zero order. The asymmetry is contained in the second term and is not affected by the introduction of terms in ρ; hence, there is consistency with the results of the preliminary theory, in which for $\rho = 0$ and even values of n, an asymmetric spectrum is obtained. It is also seen that the symmetry is re-established when the phase angle $\delta = (k + \frac{1}{2})\pi$, confirming the result of Murty and the conclusion of Section 4.

Although the series (32) and (33) are cut off after the third terms, limiting their practical use to values of $\xi < 1$, it is possible to obtain some properties from them, which agree qualitatively with the experiment. For $m = 0$ they show that the intensity of the order -1 is stronger than the intensity of the order $+1$, and that the intensity of the order $+2$ is stronger than the intensity of the order -2, facts that are confirmed qualitatively by the experiments of Ramachandra Rao.[8]

Taking for instance $a_2 = 1$, the same features may be established for $m = 1$; for $m = 2$, however, $I_5 = I_{-5}$ (in the considered approximation) and from then on all the intensities of positive orders should be stronger than the intensities of corresponding negative orders.

It has already been remarked by Ramachandra Rao that the nature of the asymmetry in the case of parallel sound waves for even values of n is entirely different from the asymmetry caused by the oblique incidence of the light, in which the intensity of any positive order is greater than the intensity of the corresponding negative order.

7. Explicit Calculation of the Intensities by the Series Method in the Case $n = 3$

Taking into account the form of the series (23), we put for $n = 3$,

$$\left.\begin{aligned}
\phi_0 &= \sum_{k=0}^{\infty} A_{0k}\xi^k \\
\phi_{\pm 3m \pm 1} &= \frac{\xi^{m+1} a_3^{\,m}}{2^{m+1} m!} \sum_{k=0}^{\infty} A_{\pm 3m \pm 1, k}\xi^k \\
\phi_{\pm 3m \pm 2} &= \frac{\xi^{m+2} a_3^{\,m+1}}{2^{m+2} (m+1)!} \sum_{k=0}^{\infty} A_{\pm 3m \pm 2, k}\xi^k \\
\phi_{\pm 3m \pm 3} &= \frac{\xi^{m+1} a_3^{\,m+1}}{2^{m+1} (m+1)!} \sum_{k=0}^{\infty} A_{\pm 3m \pm 3, k}\xi^k
\end{aligned}\right\} \qquad (34)$$

The factors preceding the series have again been chosen as the corresponding lowest power coefficients of the series expansion of the solution (17) in the case $\rho = 0$, excepted for the factor of $\phi_{\pm 3m \pm 2}$, which should read

$$\left(\frac{m+1}{a_3} - 2e^{\mp i\delta}\right) \frac{\xi^{m+2} a_3^{\,m+1}}{2^{m+3} (m+1)!},$$

but which has been chosen in the above form in order to simplify the relations between the series coefficients. Substitution of the series (34) into the equation (9) leads to the following relations between the coefficients A_{rk},

$$(m + k + 2) A_{\pm 3m \pm 1, k+1} \mp A_{\pm 3m, k+1} \pm \frac{a_3}{4(m+1)} A_{\pm 3m \pm 2, k-1}$$

$$\mp m A_{\pm 3m \mp 2, k+1} e^{\mp i\delta} \pm \frac{a_3^2}{4(m+1)} A_{\pm 3m \pm 4, k-1} e^{\pm i\delta}$$

$$= \frac{i\rho}{2} (3m + 1)^2 A_{\pm 3m \pm 1, k} \qquad (m \geqslant 1) \tag{35}$$

$$(k + 2) A_{\pm 1, k+1} \mp A_{0, k+1} \pm \frac{a_3}{4} A_{\pm 2, k-1} \mp \frac{a_3^2}{4} A_{\mp 2, k-1} e^{\mp i\delta}$$

$$\pm \frac{a_3^2}{4} A_{\pm 4, k-1} e^{\pm i\delta}$$

$$= \frac{i\rho}{2} A_{\pm 1, k} \tag{36}$$

$$(m + k + 3) A_{\pm 3m \pm 2, k+1} \mp \frac{m+1}{a_3} A_{\pm 3m \pm 1, k+1} \pm A_{\pm 3m \pm 3, k+1}$$

$$\mp (m + 1) A_{\pm 3m \mp 1} e^{\mp i\delta} \pm \frac{a_3^2}{4(m+2)} A_{\pm 3m \pm 5, k-1} e^{\pm i\delta}$$

$$= \frac{i\rho}{2} (3m + 2)^2 A_{\pm 3m \pm 2, k} \qquad (m \geqslant 1) \tag{37}$$

$$(k + 3) A_{\pm 2, k+1} \mp \frac{1}{a_3} A_{\pm 1, k+1} \pm A_{\pm 3, k+1} \mp A_{\mp 1, k+1} e^{\mp i\delta}$$

$$\pm \frac{a_3^2}{8} A_{\pm 5, k-1} e^{\pm i\delta}$$

$$= 2i\rho A_{\pm 2, k} \tag{38}$$

$$(m + k + 2) A_{\pm 3m \pm 3, k+1} \mp \tfrac{1}{4} A_{\pm 3m \pm 2, k-1} \pm \tfrac{1}{4} A_{\pm 3m \pm 4, k-1}$$

$$\mp (m + 1) A_{\pm 3m, k+1} e^{\mp i\delta} \pm \frac{a_3^2}{4(m+2)} A_{\pm 3m \pm 6, k-1} e^{\pm i\delta}$$

$$= \frac{i\rho}{2} (3m + 3)^2 A_{\pm 3m \pm 3, k} \qquad (m \geqslant 0) \tag{39}$$

$$(k + 1) A_{0, k+1} - \tfrac{1}{4} A_{-1, k-1} + \tfrac{1}{4} A_{1, k-1}$$

$$- \frac{a_3^2}{4} A_{-3, k-1} e^{-i\delta} + \frac{a_3^2}{4} A_{3, k-1} e^{i\delta} = 0. \tag{40}$$

In order to transform the formulæ with the upper signs into the formulæ with the lower signs the substitution:

$$\left. \begin{array}{lll} A_{rk} & \to & (-1)^r A_{-r,k} \\ \delta & \to & -\delta \end{array} \right\} \tag{41}$$

must be performed. The same transformation changes the formulæ for A_{rk} into the expressions for $A_{-r,k}$.

From the relations (35)-(40) one obtains,

$$A_{3m+3,\,0} = e^{-i(m+1)\delta} \tag{42 a}$$

$$A_{3m+1,\,0} = e^{-im\delta} \tag{42 b}$$

$$A_{3m+2,\,0} = \left(\frac{m+1}{2a_3} - e^{-i\delta}\right) e^{-im\delta} \tag{42 c}$$

$$A_{3m+3,\,1} = \tfrac{3}{4}\, i\rho\,(m+1)(2m+3)\, e^{-i(m+1)\delta} \tag{42 d}$$

$$A_{3m+1,\,1} = \frac{i\rho}{4}(m+1)(6m+1)\, e^{-im\delta} \tag{42 e}$$

$$A_{3m+2,\,1} = \frac{i\rho}{24a_3}(m+1)(18m^2+33m+10)\, e^{-im\delta} - \frac{i\rho}{4}(6m^2+11m+6)\, e^{-i(m+1)\delta} \tag{42 f}$$

$$A_{3m+3,\,2} = \frac{m+1}{24a_3} e^{-im\delta} - \frac{1}{4} e^{-i(m+1)\delta} - \frac{a_3^2}{4(m+2)} e^{-i(m+1)\delta} - \frac{9}{160}\rho^2 (m+1)(2m+3)(2m+5) \times (5m+4)\, e^{-i(m+1)\delta} \tag{42 g}$$

$$A_{3m+1,\,2} = \frac{m}{96a_3} e^{-i(m-1)\delta} - \frac{1}{8} e^{-im\delta} + \frac{a_3}{8(m+1)} e^{-i(m+1)\delta} - \frac{a_3^2}{4(m+1)} e^{-im\delta} - \frac{\rho^2}{480}(540m^4+1692m^3 + 1305m^2 + 213m + 20)\, e^{-im\delta} \tag{42 h}$$

$$A_{3m+3,\,3} = \frac{i\rho}{96a_3}(m+1)(6m^2+15m+7)\, e^{-im\delta} + \text{terms in } i\rho\, e^{-i(m+1)\delta} \tag{42 i}$$

$$A_{3m+1,3} = \frac{i\rho a_3}{96}\frac{18m^2+21m+13}{m+1}e^{-i(m+1)\delta}$$
$$+\frac{i\rho}{384a_3}m(6m^2+7m-1)e^{-i(m-1)\delta}$$
$$+\text{ terms in } i\rho\, e^{-im\delta} \qquad (42\,j)$$

In (42 i) and (42 j) only those terms are written that give a contribution to the term in ξ^3 in the series for the intensities.

We have finally calculated the expressions for the intensities as far the first term of the series contributing to the asymmetry. We find

$$\mathrm{I}_{\pm 3m\pm 1} = \frac{\xi^{2m+2}a_3^{2m}}{2^{2m+2}(m\,!)^2}\Big\{1+\xi^2\Big[-\frac{1}{4}-\frac{a_3^2}{2(m+1)}$$
$$+\frac{1}{4}\Big(\frac{m}{12a_3}+\frac{a_3}{m+1}\Big)\cos\delta$$
$$-\frac{\rho^2}{240}(432m^3+390m^2+3m+5)\Big]$$
$$\pm\,\xi^3\frac{\rho}{24}\Big(\frac{m}{4a_3}+\frac{5a_3}{m+1}\Big)\sin\delta+\dots\Big\} \qquad (43)$$

$$\mathrm{I}_{\pm 3m\pm 2} = \frac{\xi^{2m+4}a_3^{2m+2}}{2^{2m+4}[(m+1)!]^2}\Big\{1+\frac{(m+1)^2}{4a_3^2}-\frac{m+1}{a_3}\cos\delta$$
$$\mp\,\xi\frac{2}{3}\rho\frac{m+1}{a_3}\sin\delta+\dots\Big\} \qquad (44)$$

$$\mathrm{I}_{\pm 3m\pm 3} = \frac{\xi^{2m+2}a_3^{2m+2}}{2^{2m+2}[(m+1)!]^2}\Big\{1-\frac{\xi^2}{2}\Big[1+\frac{a_3^2}{m+2}$$
$$-\frac{m+1}{6a_3}\cos\delta+\frac{9}{40}\rho^2(m+1)(2m+3)$$
$$\times(8m+5)\Big]\pm\xi^3\frac{\rho}{24a_3}(m+1)\sin\delta+\dots\Big\}. \qquad (45)$$

The intensities of those spectra also show an asymmetry with respect to the zero order line. The asymmetry disappears when ρ is put equal to zero, thus confirming the results of the preliminary theory, or when the phase angle $\delta = k\pi$ in accordance with the conclusions of Section 4.

From the formulæ it is to be expected that at least for values of $\xi < 1$, the intensities of the orders $3m+1$ and $3m+3$ should respectively be greater than the intensities of the orders $-3m-1$ and $-3m-3$, and that the intensity of the order $-3m-2$ should be stronger than the intensity of the order $3m+2$. Here also we have a marked difference with the nature of the asymmetry in the case of oblique incidence of the light.

With regard to the lowest power term in ξ we may also notice that for small values of ξ the intensities of orders $\pm 3m \pm 2$ should be much weaker than the intensities of the orders $\pm 3m \pm 1$ and $\pm 3m \pm 3$.

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Summary

The theory of the diffraction of light by two superposed parallel supersonic waves, consisting of the fundamental tone and the n-th harmonic is developed, starting from the wave equation for the electric field of the light. The Fourier series method, first used in Raman and Nath's generalized theory, is here employed to derive a system of difference-differential equations for the amplitudes of the diffracted light waves. The r-th order spectrum makes an angle $\theta_r = -Arc \sin r\,\lambda/\lambda^*$ with the direction of the incident light and presents a change of frequency $-r\nu^*$. In the case that the right-hand side of the difference-differential equations may be neglected, the exact solution is obtained by means of a complex function method. From the structure of the general system of difference-differential equations it is shown that the intensities of the orders r and $-r$ are always different for an even as well as for an odd ratio of the sound frequencies, excepted for some special values of the phase angle of the sound waves. A solution of the general system for $n=2$ is written in the form of a power series in ξ, the terms of which are calculated till the third ones included; the asymmetry of the intensities of opposite orders is not due to the terms in ρ. In the case $n=3$ a series solution also leads to an asymmetric pattern with respect to the zero order line; the terms in ρ are here responsible for the asymmetry, so that the symmetry property reappears for $\rho=0$, in accordance with the simplified theory based on Raman and Nath's preliminary theory.

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