SELF-RECIPROCAL FUNCTIONS IN THE FORM OF SERIES

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1. THE object of this note is to find out a few Self-Reciprocal functions in the form of series. Following the usual notation, I denote a function $f(x)$ as R_μ, if it is Self-Reciprocal for Hankel Transforms of order μ so that it is given by.

$$f(x) = \int_0^\infty J_\mu(xy) f(y) \sqrt{xy}\, dy, \tag{1.1}$$

where $J_\mu(x)$ is a Bessel function of order μ. If $\mu = -\frac{1}{2}$, $f(x)$ is denoted as R_c while for $\mu = \frac{1}{2}$, $f(x)$ is written as R_s.

2. Dr. Willis[3] has obtained expansions for certain Integrals, in the form of infinite series. Further Dr. Brij Mohan[1] has obtained a class of kernels for Self-Reciprocal functions of Hankel Transforms. On applying these kernels in the Integrals given by Dr. Willis, we find that the expansions in series become Self-Reciprocal functions. Accordingly we show in the following lines that certain Self-Reciprocal functions can be represented as infinite series.

3. Dr. Willis[3] has shown that

$$\int_0^\infty f(x)\, e^{-mx} dx = \frac{1}{m}\left\{f(0) + \frac{f^{(1)}(0)}{m} + \frac{f^{(2)}(0)}{m^2} + \dots\right\}. \tag{3.1}$$

Let $f(x)$ be R_c or R_s. Then according to Dr. Brij Mohan[1] the kernel

$$\bar{e}^{x}, \tag{3.2}$$

transforms

$$R_c\,(R_s) \text{ into } R_s\,(R_c)$$

Hence we find that

$$\int_0^\infty \bar{e}^{mx} f(x)\, dx, \tag{3.3}$$

is $R_s\,(R_c)$ according as $f(x)$ is $R_c\,(R_s)$.

Hence from (3.3) and (3.1), we conclude that

$$\frac{1}{x}\left\{f(0)+\frac{f^{(1)}(0)}{x}+\frac{f^{(2)}(0)}{x^2}+\ldots.\right\}. \tag{3.4}$$

is $R_s\,(R_c)$ according as $f(x)$ is $R_c\,(R_s)$.

4. Again, Dr. Willis[3] has shown that

$$\int_0^\infty f(y)\sin xy\,dy=\frac{1}{x}\left\{f(0)+\frac{f^{(1)}(0)}{x^2}+\frac{f^{(2)}(0)}{x^4}+\ldots.\right\}. \tag{4.1}$$

Let $f(x)$ be R_s. Then by the definition of an R_s function we find that

$$\int_0^\infty f(y)\sin xy\,dy=f(x). \tag{4.2}$$

Hence from (4.2) and (4.1), we conclude that

$$\frac{1}{x}\left\{f(0)+\frac{f^{(1)}(0)}{x^2}+\frac{f^{(2)}(0)}{x^4}+\ldots.\right\}, \tag{4.3}$$

is R_s if $f(x)$ is R_s.

5. Further, it has been shown by Dr. Willis[3] that

$$\int_0^\infty f(y)\,J_0(xy)\,dy=\frac{1}{x}\left\{f(0)-\frac{f^{(2)}(0)}{2\angle 1\,x^2}+\frac{1\cdot 3f^{(4)}(0)}{2^2\angle 2\,x^4}-\frac{1.3.5f^{(6)}(0)}{2^3\angle 3\,x^6}+\ldots.\right\}. \tag{5.1}$$

Let $f(x)$ be R_c or R_s.

Dr. Brij Mohan[1] has shown that the kernel

$$J_0(x), \tag{5.2}$$

transforms

$R_c\,(R_s)$ into $R_s\,(R_c)$.

Hence from (5.1) and (5.2), we conclude that

$$\frac{1}{x}\left\{f(0)-\frac{f^{(2)}(0)}{2\angle 1\,x^2}+\frac{1\cdot 3f^{(4)}(0)}{2^2\angle 2\,x^4}-\frac{1.3.5.f^{(6)}(0)}{2^3\angle 3\,x^6}+\ldots.\right\}, \tag{5.3}$$

is $R_s\,(R_c)$ according as $f(x)$ is $R_c\,(R_s)$.

6. It has been further shown by Dr. Willis[3] that

$$\int_0^\infty f(x)\, J_1(xy)\, dy = \frac{f(0)}{x} + \frac{1}{x}\left\{\frac{f^{(1)}(0)}{x} + \frac{f^{(3)}(0)}{2\,\angle 1\, x^3} + \frac{1.3 f^{(5)}(0)}{2^2\,\angle 2\, x^5} - \frac{1.3.5 f^{(7)}(0)}{2^3\,\angle 3\, x^7} + \ldots\right\}. \tag{6.1}$$

Dr. Brij Mohan[1] has shown that the kernel

$$x^{\frac{1}{2}(\mu-\nu+1)}\, J_{\frac{1}{2}(\mu+\nu)}(x), \tag{6.2}$$

transforms R_μ into R_ν.

Putting $\nu = (\mu + 1)$ we find that the kernel

$$J_{(\mu+\frac{1}{2})}(x), \tag{6.3}$$

transforms R_μ into $R_{(\mu+1)}$; and in particular we find that

$$J_1(x), \tag{6.4}$$

transforms R_s into $R_{3/2}$.

Therefore, we find that

$$\int_0^\infty f(y)\, J_1(xy)\, dy, \tag{6.5}$$

is $R_{3/2}$ if $f(x)$ is R_s.

Hence from (6.5) and (6.1), we further conclude that

$$\frac{f(0)}{x} + \frac{1}{x}\left\{\frac{f^{(1)}(0)}{x} - \frac{f^{(3)}(0)}{2\,\angle 1\, x^3} + \frac{1.3 f^{(5)}(0)}{2^2\,\angle 2\, x^5} - \frac{1.3.5 f^{(7)}(0)}{2^3\,\angle 3\, x^7} + \ldots\right\}, \tag{6.6}$$

is $R_{3/2}$ if $f(x)$ is R_s.

7. Again Dr. Willis[3] has also shown that

$$\int_0^\infty f(y)\, \frac{\sin xy}{y}\, dy = \frac{\pi}{2} f(0) + \left\{\frac{f^{(1)}(0)}{x} - \frac{f^{(3)}(0)}{3x^3} + \frac{f^{(5)}(0)}{5x^5} - \ldots\right\}, \tag{7.1}$$

so that

$$\int_0^\infty f(y)\frac{\sin xy}{xy}dy = \frac{1}{x}\left[\frac{\pi}{2}f(0) + \left\{\frac{f^{(1)}(0)}{x} - \frac{f^{(3)}(0)}{3x^3} + \frac{f^{5}(0)}{5x^5} - \ldots\right\}\right]. \tag{7.2}$$

Dr. Brij Mohan[1] has shown that the kernel

$$\frac{\sin x}{x}, \tag{7.3}$$

transforms $R_c\,(R_{3/2})$ into $R_{3/2}\,(R_c)$.

Hence it follows from (7.2) and (7.3) that

$$\frac{1}{x}\left[\frac{\pi}{2}f(0) + \left\{\frac{f^{(1)}(0)}{x} - \frac{f^{(3)}(0)}{x^3} + \frac{f^{(5)}(0)}{5x^5} - \ldots\right\}\right], \tag{7.4}$$

is $R_{3/2}\,(R_c)$ according as $R_c\,(R_{3/2})$.

8. It has been further shown by Dr. Willis[3] that

$$\int_0^\infty f(y)e^{-y^2x^2}dy = \frac{\sqrt{\pi}}{2x}\left\{f(0) + \frac{1}{\angle 2}\cdot\frac{1}{2}\frac{f^{(2)}(0)}{x^2} + \frac{1}{\angle 4}\frac{1.3}{2^2}\frac{f^{(4)}(0)}{x^4} + \ldots\right\} + \frac{2}{x}\left\{\frac{f^{(1)}(0)}{x} + \frac{\angle 1}{\angle 3}\frac{f^{(3)}(0)}{x^3} + \frac{\angle 2}{\angle 5}\cdot\frac{f^{(5)}(0)}{x^5} + \ldots\right\} \tag{8.1}$$

Putting x for x^2 and y for y^2 we find that

$$\int_0^\infty \frac{f(\sqrt{y})}{\sqrt{y}}\,e^{-xy}dy = \sqrt{\frac{\pi}{x}}\left\{f(0) + \frac{1}{\angle 2}\cdot\frac{1}{2}\frac{f^{(2)}(0)}{x} + \frac{1}{\angle 4}\cdot\frac{1}{2^2}\frac{f^{(4)}(0)}{x^2} + \ldots\right\} + 4\left\{\frac{f^{(1)}(0)}{x} + \frac{\angle 1}{\angle 3}\frac{f^{(3)}(0)}{x^2} + \frac{\angle 2}{\angle 5}\frac{f^{(5)}(0)}{x^3} + \ldots\right\}. \tag{8.2}$$

Hence from (3.2) and (8.2), it follows that the series

$$\sqrt{\frac{\pi}{x}}\left\{f(0)+\frac{1}{\angle 2}\cdot\frac{1}{2}\frac{f^{(2)}(0)}{x}+\frac{1}{\angle 4}\frac{1}{2^2}\frac{f^{(4)}(0)}{x^2}+\ldots\right\}$$

$$+4\left\{\frac{f^{(1)}(0)}{x}+\frac{\angle 1}{\angle 3}\frac{f^{(3)}(0)}{x^2}+\frac{\angle 2}{\angle 5}\frac{f^{(5)}(0)}{x^3}\right.$$

$$\left.+\ldots\right\}, \qquad (8.3)$$

is R_c (R_s) if the function

$$\frac{f(\sqrt{y})}{\sqrt{y}}, \qquad (8.4)$$

is R_s (R_c). In particular if

$$f(\sqrt{y})=1, \qquad (8.5)$$

we find that (8.4) reduces to the well known function[2]

$$\frac{1}{\sqrt{y}} \qquad (8.6)$$

which is R_s (R_c). Accordingly (8.3) also reduces to the function

$$\frac{1}{\sqrt{x}}, \qquad (8.7)$$

which, then becomes R_c (R_s).

9. Finally it has also been shown by Dr. Willis[3] that the Asymptotic expansion of the function

$$\int_0^\infty \bar{e}^{y^2} J_0(xy)\,dy, \qquad (9.1)$$

is

$$\frac{1}{x}+\frac{1^2}{\angle 1}\frac{1}{x^3}+\frac{1^2.3^2}{\angle 2}\cdot\frac{1}{x^5}+\frac{1^2.3^2.5^2}{\angle 3}\frac{1}{x^7}+\ldots. \qquad (9.2)$$

Putting $\sqrt{2}x$ for x, and $y/\sqrt{2}$ for y, we find that the Asymptotic expansion of

$$\frac{1}{\sqrt{2}}\int_0^\infty \bar{e}^{y^2/2} J_0(xy)\,dy, \qquad (9.3)$$

is given by

$$\frac{1}{2^{1/2}x}+\frac{1^2}{\angle 1}\cdot\frac{1}{2^{3/2}x^3}+\frac{1^2.3^2}{\angle 2}\cdot\frac{1}{2^{5/2}x^5}+\frac{1^2.3^2.5^2}{\angle 3}\frac{1}{2^{7/2}x^7}+\ldots. \qquad (9.4)$$

Also Dr. Brij Mohan[2] has shown that

$$e^{-x^2/2}, \tag{9.5}$$

is R_c. Hence from (9.5) and (5.2) it follows that

$$g(x) = \frac{1}{\sqrt{2}} \int_0^\infty e^{-y^2/2} J_0(xy)\, dy, \tag{9.6}$$

is R_s.

Also, it has been shown by Willis[3] that

$$\int_0^\infty e^{-y^2} J_0(xy)\, dy = \sqrt{\pi}\, e^{-x^2/8} I_0\left(\frac{x^2}{8}\right),$$

so that

$$\int_0^\infty e^{-y^2/2} J_0(xy)\, dy = \sqrt{2\pi}\, e^{-x^2/4} I_0\left(\frac{x^2}{4}\right). \tag{9.7}$$

Hence from (9.7), (9.6) and (9.4), we conclude that the R_s function

$$e^{\frac{-x^2}{4}} I_0\left(\frac{x^2}{4}\right) \tag{9.8}$$

has its Asymptotic expansion given by the series

$$\sqrt{\frac{2}{\pi}}\left[\frac{1}{2x} + \frac{1^2}{\angle 1}\frac{1}{2^2x^3} + \frac{1^2.3^2}{\angle 2}\frac{1}{2^3x^5} + \frac{1^2.3^2.5^2}{\angle 3}\frac{1}{2^4x^7} + \ldots\right]. \tag{9.9}$$

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