ON EQUAL SUMS OF LIKE POWERS

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I

LET $A_{i\,(=1,\,2,\,3,\,4)}$ and B_i be eight integers such that

$$\sum_{i=1}^{4} A_i^n = \sum_{i=1}^{4} B_i^n \qquad (n = 1, 3, 5). \tag{1}$$

Then, if we put

$$\begin{aligned} A_1 &= x - y,\ A_2 = y + w,\ A_3 = z - v,\ A_4 = v + u, \\ B_1 &= x + y,\ B_2 = w - y,\ B_3 = z + v,\ B_4 = u - v, \end{aligned} \tag{2}$$

we have to determine x, y, z, u, v and w so as to satisfy the Diophantine system

$$\begin{aligned} v\,(u^2 - x^2) = y\,(x^2 - w^2)\,(2v)^2 &+ (u + z)^2 + (u - z)^2 \\ &= (2y)^2 + (x + w)^2 + (x - w)^2. \end{aligned} \tag{3}$$

Now, by means of Goldbach's identity

$$\begin{aligned} p^2 + q^2 - (3r - p - q)^3 = (2r - p)^2 &+ (2r - q)^2 \\ &+ (r - p - q)^2, \end{aligned} \tag{4}$$

we see that Eq. (3_2) is satisfied if we have

$$\begin{aligned} &v = \tfrac{1}{2} p,\ u + z = 3r - p - q,\ u - z = q \\ &y = \tfrac{1}{2}(2r - p),\quad x + w = 2r - q,\quad x - w = r - p - q. \end{aligned} \tag{5}$$

Hence we get

$$\begin{aligned} u &= \tfrac{1}{2}(3r - p) \\ z = x &= \tfrac{1}{2}(3r - p - 2q) \\ w &= \tfrac{1}{2}(r + p) \end{aligned}$$

and the first of Eq. (3) takes the form

$$2r^2 - 3\,(p + q)\,r + (p^2 + pq + q^2) = 0. \tag{6}$$

Therefore,

$$r = \frac{3(p+q) \pm \sqrt{p^2 + 10pq + q^2}}{4}$$

and, to exclude the case $r =$ irrational, it is necessary and sufficient to take

$$p^2 + 10pq + q^2 = t^2.$$

From this we find

$$p = 2ab + 10b^2, \quad q = a^2 - b^2, \quad t = a^2 + 10ab + b^2$$

and then we have

$$r_1 = a^2 + 4ab + 7b^2, \qquad r_2 = \tfrac{1}{2}(a^2 - 2ab + 13b^2).$$

Substituting the foregoing values in the Eqs. (5_1), (5_4) and (6) we get

$$\begin{array}{r|r} x_1 = z_1 = \tfrac{1}{2}(a^2 + 10ab + 13b^2) & x_2 = z_2 = \tfrac{1}{4}(-a^2 - 10ab + 23b^2) \\ y_1 = a^2 + 3ab + 2b^2 & y_2 = \tfrac{1}{2}(a^2 - 4ab + 3b^2) \\ w_1 = \tfrac{1}{2}(a^2 + 6ab + 17b^2) & w_2 = \tfrac{1}{2}(a^2 + 2ab + 33b^2) \\ v_1 = ab + 5b^2 & v_2 = ab + 5b^2 \\ u_1 = \tfrac{1}{2}(3a^2 + 10ab + 11b^2) & u_2 = \tfrac{1}{4}(3a^2 - 10ab + 19b^2). \end{array}$$

Thus, from (2) and (1), we see that either

$$\begin{aligned} & -a^2 + 4ab + 9b^2,\ 3a^2 + 12ab + 21b^2,\ a^2 + 8ab + 3b^2, \\ & \quad 3a^2 + 12ab + 21b^2 \\ & \overset{n}{=} 3a^2 + 16ab + 17b^2,\ -a^2 + 13b^2,\ a^2 + 12ab + 23b^2, \\ & \quad 3a^2 + 8ab + b^2, \text{ (where } n = 1, 3, 5), \end{aligned} \tag{7*}$$

or

$$\begin{aligned} & -3a^2 - 2ab + 17b^2,\ 3a^2 - 6ab + 39b^2,\ -a^2 - 14ab + 3b^2, \\ & \quad 3a^2 - 6ab + 39b^2 \\ & \overset{n}{=} a^2 - 18ab + 29b^2,\ -a^2 + 10ab + 27b^2, \\ & \quad -a^2 - 6ab + 43b^2,\ 3a^2 - 14ab - b^2 \quad (n = 1, 3, 5). \end{aligned} \tag{8}$$

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* The separation of two sets of numbers by the symbol $\overset{n}{=}$ denotes that they have the same sums of kth power for $k = 1, 2, .. n$, or for noted values of n.

Hence we have a process for finding solutions of the system (1) and in the case: $A_4 = A_2$, $A_1 + A_2 = B_1 + B_2$.

Also, by means of (3), we find that either

$$\left.\begin{aligned}
&(2ab + 10b^2)(2a^2 + 10ab + 12b^2)(a^2 - b^2) \\
&\qquad = (2a^2 + 6ab + 4b^2)(a^2 + 8ab + 15b^2) \\
&\qquad\quad \times (2ab - 2b^2) \\
&(2ab + 10b^2)^2 + (2a^2 + 10ab + 12b^2)^2 + (a^2 - b^2)^2 \\
&\qquad = (2a^2 + 6ab + 4b^2)^2 + (a^2 + 8ab + 15b^2)^2 \\
&\qquad\quad + (2ab - 2b^2)^2
\end{aligned}\right\} \tag{9}$$

or

$$\left.\begin{aligned}
&(2ab + 10b^2)\left(\frac{a^2 - 10ab + 21b^2}{2}\right)(a^2 - b^2) \\
&\qquad = (a^2 + 4ab + 3b^2)(- 2ab + 14b^2) \\
&\qquad\quad \times \left(\frac{- a^2 - 6ab - 5b^2}{2}\right) \\
&(2ab + 10b^2)^2 + \left(\frac{a^2 - 10ab + 21b^2}{2}\right)^2 + (a^2 - b^2)^2 \\
&\qquad = (a^2 - 4ab + 3b^2)^2 + (- 2ab + 14b^2)^2 \\
&\qquad\quad + \left(\frac{- a^2 - 6ab - 5b^2}{2}\right)^2.
\end{aligned}\right\}. \tag{10}$$

It is easy to see that the equalities (9) express the condition that two rectangular parallelopipeds shall have integral edges, equal volumes and equal diagonals, if all the factors are positive.

Though the linear substitutions $a = a_1 - 5b_1$, $b = 2b_1$ transform, finally, the relation (7) into (8), however we consider them both, in order to find some other results. Also we consider (9) and (10).

II

From (4) we get the double relation

$$p,\ q,\ 3r - p - q \stackrel{2}{=} 2r - p,\ 2r - q,\ p + q - r.$$

Hence (9_2) and (10_2) become

$$\begin{aligned}
&2ab + 10b^2,\ 2a^2 + 10ab + 12b^2,\ a^2 - b^2 \\
&\qquad \stackrel{2}{=} 2a^2 + 6ab + 4b^2,\ a^2 + 8ab + 15b^2,\ 2b^2 - 2ab
\end{aligned}$$

$$a^2 - 4ab + 3b^2,\ -2ab + 14b^2,\ \frac{a^2 + 6ab + 5b^2}{2}$$

$$\stackrel{2}{=} 2ab + 10b^2,\ \frac{a^2 - 10ab + 21b^2}{2},\ a^2 - b^2,$$

from which we find

$$2a^2+10ab+12b^2,\ a^2-4ab+3b^2,\ -2ab+14b^2,\ \frac{a^2+6ab+5b^2}{2}$$

$$\stackrel{2}{=} 2a^2 + 6ab + 4b^2,\ a^2 + 8ab + 15b^2,\ -2ab + 2b^2,$$

$$\frac{a^2 - 10ab + 21b^2}{2}.$$

On the other hand, from (9_1) and (10_1), we take

$$(2a^2+10ab+12b^2)\,(a^2-4ab+3b^2)\,(-2ab+14b^2)\,\frac{(a^2+6ab+5b^2)}{2}$$

$$= (2a^2 + 6ab + 4b^2)\,(a^2 + 8ab + 15b^2)\,(-2ab + 2b^2)$$

$$\times \left(\frac{a^2 - 10ab + 21b^2}{2}\right).$$

Therefore we have a process for finding solutions of the system

$$A_1,\ A_2,\ A_3,\ A_4 \stackrel{2}{=} B_1,\ B_2,\ B_3,\ B_4$$

$$A_1A_2A_3A_4 = B_1B_2B_3B_4.$$

Thus,

$$4,\ 8,\ 30,\ 35 \stackrel{2}{=} 3,\ 14,\ 20,\ 40$$

$$4.8.30.35 = 3.14.20.40$$

and the polynomials

$$x^4 - 77x^3 + 1862x^2 - 14680x + 33600,$$

$$x^4 + 77x^3 + 1862x^2 - 16120x + 33600$$

have respectively the roots (4, 8, 30, 35) and (3, 14, 20, 40).

III

A two-parameter solution of the system

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= z_1 + z_2 + z_3 + z_4 \\ x_1 + x_4 = x_2 + x_3 &= z_1 + z_4 = z_2 + z_3 \qquad (11) \\ x_1x_2x_3x_4 &= z_1z_2z_3z_4 \end{aligned}$$

may be found from (7).

In fact, for $A_1 + A_2 = B_1 + B_2$, $A_3 + A_4 = B_3 + B_4$, we find that every solution of (1) satisfies the system

$$(A_1 + A_3) + (A_1 + A_4) + (A_2 + A_3) + (A_2 + A_4)$$
$$= (B_1 + B_3) + (B_1 + B_4) + (B_2 + B_3) + (B_2 + B_4)$$
$$(A_1 + A_3)(A_1 + A_4)(A_2 + A_3)(A_2 + A_4)$$
$$= (B_1 + B_3)(B_1 + B_4)(B_2 + B_3)(B_2 + B_4).$$

Therefore, from (7), we get the following identities

$$(6ab + 6b^2) + (a^2 + 8ab + 15b^2) + (2a^2 + 10ab + 12b^2)$$
$$+ (3a^2 + 12ab + 21b^2)$$
$$= (2a^2 + 14ab + 20b^2) + (3a^2 + 12ab + 9b^2)$$
$$+ (6ab + 18b^2) + (a^2 + 4ab + 7b^2)$$

and

$$(6ab + 6b^2)(a^2 + 8ab + 15b^2)(2a^2 + 10ab + 12b^2)$$
$$\times (3a^2 + 12ab + 21b^2)$$
$$= (2a^2 + 14ab + 20b^2)(3a^2 + 12ab + 9b^2)(6ab + 18b^2)$$
$$\times (a^2 + 4ab + 7b^2),$$

by means of which we take particular solutions of the system (11).

Thus,

$$6 + 12 + 15 + 21 = 7 + 9 + 18 + 20$$
$$6 + 21 = 12 + 15 = 7 + 20 = 9 + 18.$$
$$6.12.15.21 = 7.9.18.20.$$

IV

The relation (8), for $a = -a_0 + 4b_0$, $b = a_0$, may be finally written

$$a_0^2 + a_0b_0 - 3b_0^2,\ 3(a_0^2 - a_0b_0 + b_0^2),\ a_0^2 - 3a_0b_0 - b_0^2,$$
$$\times 3(a_0^2 - a_0b_0 + b_0^2)$$
$$\stackrel{n}{=} 3a_0^2 - 5a_0b_0 + b_0^2, a_0^2 + 3a_0b_0 - b_0^2, 3a_0^2 - a_0b_0 - b_0^2,$$
$$a_0^2 - 5a_0b_0 + 3b_0^2, \qquad (n = 1,\ 3,\ 5)$$

whilst (7), for $a = a_0 + b_0$, $b = a_0 - b_0$, gives

$$3a_0^2 - 5a_0b_0 + b_0^2,\ 3(3a_0^2 - 3a_0b_0 + b_0^2),\ 3a_0^2 - a_0b_0 - b_0^2,$$
$$3(3a_0^2 - 3a_0b_0 + b_0^2)$$

$$\stackrel{n}{=} 9a_0^2 - 7a_0b_0 + b_0^2, 3a_0^2 - 7a_0b_0 + 3b_0^2, 9a_0^2 - 11a_0b_0 + 3b_0^2, 3a_0^2 + a_0b_0 - b_0^2 \qquad (n = 1, 3, 5).$$

Adding the foregoing two relations, we take, after the omission of the index 0, the new relation

$$a^2 - 3ab - b^2, \ a^2 + ab - 3b^2, \ 3(a^2 - ab + b^2), \ 3(a^2 - ab + b^2), 3(3a^2 - 3ab + b^2), 3(3a^2 - 3ab + b^2)$$
$$\stackrel{n}{=} a^2 + 3ab - b^2, a^2 - 5ab + 3b^2, 3a^2 - 7ab + 3b^2, 3a^2 + ab - b^2, \ 9a^2 - 11ab + 3b^2, \ 9a^2 - 7ab + b^2$$
$$(n = 1, 3, 5), \qquad (12)$$

by means of which we get particular solutions of the system

$$A_1, A_2, A_3, A_3, A_4, A_4 \stackrel{n}{=} B_1, B_2, B_3, B_4, B_5, B_6 \quad (n = 1, 3, 5).$$

Thus, for $a = 2$, $b = -1$, we have

$$9, (-1), 21, 21, 57, 57 \stackrel{n}{=} -3, 17, 29, 9, 51, 61, \qquad (n = 1, 3, 5)$$

that is

$$3, 21, 21, 57, 57 \stackrel{n}{=} 1, 17, 29, 51, 61. \qquad (n = 1, 3, 5)$$

V

By interchanging the role of a and b we have, from (12),

$$-a^2 + 3ab + b^2, \ 3a^2 - 5ab + b^2, \ 3a^2 - 7ab + 3b^2, - a^2 + ab + 3b^2, \ 3a^2 - 11ab + 9b^2, \ a^2 - 7ab + 9b^2$$
$$\stackrel{n}{=} - a^2 - 3ab + b^2, \ -3a^2 + ab + b^2, \ 3(a^2 - ab + b^2), 3(a^2 - ab + b^2), \ 3(a^2 - 3ab + 3b^2), 3(a^2 - 3ab + 3b^2) \qquad (n = 1, 3, 5)$$

Adding the above relation to (12) we take

$$3(3a^2 - 3ab + b^2), \ 3(3a^2 - 3ab + b^2), \ a^2 + ab - 3b^2, a^2 - 7ab + 9b^2, \ 3a^2 - 11ab + 9b^2, \ - a^2 + ab + 3b^2, 3a^2 - 5ab + b^2$$

$$\stackrel{n}{=} 3(a^2 - 3ab + 3b^2),\ 3(a^2 - 3ab + 3b^2),\ -3a^2+ab+b^2,$$
$$9a^2 - 7ab + b^2,\ 9a^2 - 11ab + 3b^2,\ 3a^2 + ab - b^2,$$
$$a^2 - 5ab + 3b^2 \qquad (n = 1, 3, 5).$$

This affords a solution of the system

$$A, A, A_1, A_2, \ldots\ldots, A_5 \stackrel{n}{=} B, B, B_1, B_2, \ldots\ldots\ldots, B_5 \ (n = 1, 3, 5),$$

depending upon two parameters.

VI

In order to obtain solutions of the system

$$A_1, A_2, A_3 \stackrel{n}{=} B_1, B_2, B_3, B_4 \qquad (n = 1, 3, 5) \qquad (13)$$

we may proceed as follows:

Starting with the system

$$M_1,\ M_2,\ M_3,\ M_4 \stackrel{3}{=} N_1,\ N_2,\ N_3,\ N_4 \qquad (14)$$

and putting

$$\tfrac{1}{4}(M_1 + M_2 + M_3 + M_4) = H$$

we may write the relation

$$M_1 - H,\ M_2 - H,\ M_3 - H,\ M_4 - H$$
$$\stackrel{m}{=} N_1 - H,\ N_2 - H,\ N_3 - H,\ N_4 - H,$$
$$(m = 1,\ 2,\ 3,\ 5), \qquad (15)$$

where the sum of the first powers of the terms of each member is equal to zero.

Now, if

$$H = N_2,$$

we get

$$N_2 - N_1,\ M_3 - N_2,\ M_4 - N_2$$
$$\stackrel{n}{=} N_3 - N_2,\ N_4 - N_2,\ N_2 - M_2,\ N_2 - M_1,$$
$$(n = 1,\ 3,\ 5).$$

If one applies this method in the particular relation

$$2,\ 45,\ 76,\ 85 \stackrel{3}{=} 1,\ 52,\ 65,\ 90,$$

in which

$$\tfrac{1}{4}(2 + 45 + 76 + 85) = 52 = N_2,$$

one finds that a solution of the system (13) is afforded by

$$52 - 1,\ 76 - 52,\ 85 - 52 \stackrel{n}{=} 65 - 52,\ 90 - 52,\ 52 - 45,$$
$$52 - 2, \qquad (n = 1,\ 3,\ 5)$$

that is

$$51,\ 24,\ 33 \stackrel{n}{=} 13,\ 38,\ 7,\ 50, \qquad (n = 1,\ 3,\ 5)$$

where

$$51 = 13 + 38.$$

Then we shall confine our attention to the system (14), under the restriction

$$N_2 = \tfrac{1}{4}(M_1 + M_2 + M_3 + M_4).$$

For this, we consider the system

$$2v,\ z + x - y,\ 2z - 2(v + x),\ z + x + y$$
$$\stackrel{3}{=} v,\ z,\ z + x,\ 2z - (v + x),$$

in order to find a solution of (14), in which the supplementary conditions

$$M_1 = 2N_1, \quad M_2 + M_4 = 2N_3$$

exist, as in the case of the above example.

From the system under consideration we get the equivalent system

$$3v^2 + 3vx + 2x^2 - z(2v + x) + y^2 = 0$$
$$zx(7x - 3z + 12v) + 2zv(3v - 2z) - 7vx(v + x)$$
$$- 2x^3 + 2(z + x)y^2 = 0,$$

from which, eliminating y^2, we find that

$$z^2 + 6x^2 + 13v^2 - 5zx - 10zv + 13xv = 0.$$

Starting from the particular solution $z = 2$, $x = -1$, $v = 1$ of the last equation, we put $z = 2r + u$, $x = -r + w$, $v = r$. Then the above equation gives r as a rational function of u, w, so that

$$z = 3u^2 - uw + 12w^2$$
$$x = -u^2 + 6uw + 3w^2$$
$$v = u^2 - 5uw + 6w^2$$

and the first equation of the equivalent system becomes

$$y^2 = u^4 + 8u^3w - 11u^2w^2 - 18uw^2. \tag{16}$$

Therefore (14) may be written

$$2u^2 - 10uw + 12w^2,\ (2u^2 + 5uw + 15w^2) - y,$$
$$6u^2 - 4uw + 6w^2,\ (2u^2 + 5uw + 15w^2) + y$$
$$\overset{3}{=} u^2 - 5uw + 6w^2,\ 3u^2 - uw + 12w^2,\ 2u^2 + 5uw + 15w^3$$
$$6u^2 - 3uw + 15w^2$$

where the integers u, w, y satisfy the Eq. (16).

Methods for resolving such equations as (16) have been given by several writers.

Hence, we conclude as follows:

$\leqslant$ Particular solutions of the system (13) are given by means of the equalities

$$2u^2 + 4uw + 6w^2,\ 3u^2 - 3uw - 6w^2,\ (-u^2 + 6uw + 3w^2) + y$$
$$\overset{n}{=} -u^2 + 6uw + 3w^2,\ 3u^2 - 2uw + 3w^2,\ u^2 + 9uw,$$
$$(w^2 - 6uw - 3w^2) + y, \qquad (n = 1,\ 3,\ 5)$$

where the integers u, w, y satisfy the equation (16), that is to say

$$y^2 = u(u - 2w)(u + w)(u + 9w).$$

Also, from (15) and under the above restriction, we get

$$-(u^2 + 9uw),\ -u^2 + 6uw + 3w^2 - y,\ 3u^2 - 3uw - 6w^2,$$
$$-u^2 + 6uw + 3w^2 + y$$
$$\overset{m}{=} -(2u^2 + 4uw + 6w^2),\ -u^2 + 6uw + 3w^2,$$
$$3u^2 - 2uw + 3w^2,$$

where

$$m = 1,\ 2,\ 3,\ 5 \gg$$

Thus, for

$$u = 2,\ w = -3,\ y = 20,$$

we get

$$38,\ -24,\ 7 \stackrel{n}{=} -13,\ 51,\ -50,\ 33,$$

or

$$24,\ 33,\ 51 \stackrel{n}{=} 7,\ 13,\ 38,\ 50 \qquad (n = 1,\ 3,\ 5)$$

and

$$50, (-33), (-24), 7 \stackrel{m}{=} -38, (-13), 51 \qquad (m = 1, 2, 3, 5)$$

For $u = 81$, $w = 40$, $y = 2079$ we get finally

$$5893,\ 6001,\ 11907 \quad 121,\ 5200,\ 6586,\ 11894, \qquad (n = 1,\ 3,\ 5) \text{ etc.}$$

Here we remark that by means of the relations

$$R_1,\ R_2,\ R_3 \stackrel{m}{=} \mathcal{U}_1,\ \mathcal{U}_2,\ \mathcal{U}_3,\ \mathcal{U}_4 \qquad (m = 1,\ 2,\ 3,\ 5)$$
$$-R_1 = R_2 + R_3,$$

we get

$$(R_1 + 0)^2 + (R_1 + R_2)^2 + (R_1 + R_3)^2$$
$$= (\mathcal{U}_1 + \mathcal{U}_2)^2 + (\mathcal{U}_1 + \mathcal{U}_3)^2 + (\mathcal{U}_1 + \mathcal{U}_4)^2$$
$$(R_1 + 0)(R_1 + R_2)\ (R_1 + R_3) = (\mathcal{U}_1+\mathcal{U}_2)\ (\mathcal{U}_1+\mathcal{U}_3)(\mathcal{U}_1+\mathcal{U}_4).$$

Thus, from

$$51, (-13),\ (-38) \stackrel{n}{=} 50,\ (-33),\ (-24),\ 7,$$

we have

$$51^2 + 38^2 + 13^2 = 17^2 + 26^2 + 57^2$$
$$51.38.13 = 17.26.57.$$

And, to finish, we note that a two-parameter solution of (13) is afforded by the values

$$A_1 = a^5 + 2a^4b - 2a^3b^2 - 5a^2b^3 - 2ab^4$$

$$A_2 = a^4b + 3a^3b^2 - 3ab^4 - b^5$$

$$A_3 = a^5 + a^4b - a^3b^2 + a^2b^3 + 3ab^4 + b^5$$

$$B_1 = a^5 + a^4b - 2a^3b^2 - 4a^2b^3 - 4ab^4 - b^5$$

$$B_2 = 2a^4b + 3a^3b^2 - a^2b^3 - ab^4$$

$$B_3 = -a^4b + 5a^2b^3 + 4ab^4 + b^5$$

$$B_4 = a^5 + 2a^4b - a^3b^2 - 4a^2b^3 - ab^4$$

(where $A_3 = B_3 + B_4$) a and b being arbitrary integers.