SOLUTION OF CAUCHY'S PROBLEM FOR THE WAVE EQUATION
\[
\left( \frac{\partial^2}{\partial t^2} - \nabla^2 + k^2 \right) \psi = 0
\]

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1. INTRODUCTION

COPSON (1956) has given a method for the solution of the Cauchy's problem for the wave equation
\[
\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0 \quad (1)
\]
in space of any odd number of spatial dimensions. In the usual method, one gets an infinite integral over the characteristic cone. Copson was able to avoid this difficulty of the divergent integral by generalising the theory of the wave equation in one spatial dimension to an odd number of spatial dimensions. In view of the physical importance of the damped wave equation
\[
\frac{\partial^2 u}{\partial t^2} - \nabla^2 u + k^2 u = 0 \quad (2)
\]
we think it is of interest to extend Copson's method to the solution of the equation (2). We make this extension below.

2. NOTATION

We take a system of rectangular Cartesian co-ordinates \( x_1, x_2, \ldots, x_{2m-1} \) in a space of \( 2m - 1 \) dimensions and write
\[
L = \frac{\partial^2}{\partial t^2} - \nabla x^2 + k^2, \quad (3)
\]
\[
\nabla x^2 = \sum_{i=1}^{2m-1} \frac{\partial^2}{\partial x_i^2}. \quad (4)
\]
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The corresponding operators in $\xi, \tau$ space will be denoted by $A$ and $\nabla_\xi^2$:

$$A = \frac{\partial^2}{\partial \tau^2} - \nabla_\xi^2 + k^2$$

$$\nabla_\xi^2 = \sum_{i=1}^{2m-1} \frac{\partial^2}{\partial \xi_i^2}.$$  

We also write

$$r^2 = (x - \xi)^2 = \sum_{i=1}^{2m-1} (x_i - \xi_i)^2$$

and

$$s = [(t - \tau)^2 - r^2]^\frac{1}{2}.$$  

3. A Lemma on the Wave Operator

We shall prove the following:

**Lemma:** If $u$ along with its partial derivatives up to order $(m - 1)$ is continuous and if

$$U(x, t) = \int \frac{J_0(ks)}{a} u(\xi, \tau) d\xi d\tau,$$

then

$$L^m U(x, t) = 2^{m-1} \pi^{m-1} \Gamma(m) U(x, t).$$  

Here $J_0(ks)$ represents the Bessel function of order zero. Denoting the instant at which the Cauchy data is given by $t_0$, the region, $D$, of integration in (9) is defined by

$$s^2 \geq 0; \quad \tau - t < 0, \quad \tau \geq t_0$$

and is bounded by a part $V'$ of the retrograde characteristic cone and by a part $V$ of the hyperplane $\tau = t_0$ (Fig. 1).

To prove the lemma, let us define a function

$$W^{(2a)}(x, t; \xi, \tau) = \frac{1}{2^a + m - 1 \pi^{m-1} \Gamma(a)} (ks)^{a-m} J_{a-m}(ks)$$

where $a$ is a complex parameter and $J_{a-m}(ks)$ is Bessel function of order $a - m$. A straightforward differentiation of (11) shows that, if $a$ is sufficiently large,

$$LW^{(2a+2)}(x, t; \xi, \tau) = k^2 W^{(2a)}(x, t; \xi, \tau).$$
We now define an integral of fractional order $2\alpha$ by

\[I^{(2\alpha)} u (x, t) = k^{2m} \int_{\Omega} W^{(2\alpha)} (x, t; \xi, \tau) u (\xi, \tau) \, d\xi \, d\tau.\]  

(13)

If we write the Bessel function as a series in powers of its argument and test the convergence of each integral separately (Riesz 1949), we can easily see that (13) converges if \(\Re \alpha > m - 1\). Moreover, all the terms in this expansion, excepting the first term, involve \(\alpha\) as a factor and therefore become zero if \(\alpha \to 0\). Considered as a function of \(\alpha\), the first term can be analytically continued into \(\Re \alpha > 0\), and its limit as \(\alpha \to +0\) is \(u (x, t)\) (Fremberg, 1945; Copson, 1947; Majumdar and Gupta, 1948, 1949). Therefore, the analytic continuation of (13) to \(\alpha = +0\) exists and

\[\lim_{\alpha \to +0} I^{(2\alpha)} u (x, t) = u (x, t).\]  

(14)

When \(\alpha\) is sufficiently large, we get, on using (12) in (13),

\[L[I^{(2\alpha+2)} u (x, t)] = k^{2} I^{(2\alpha)} u (x, t)\]  

and, hence,

\[L^{m}[I^{(2\alpha+2m)} u (x, t)] = k^{2m} I^{(2\alpha)} u (x, t).\]  

Letting \(\alpha \to +0\), we find on using (14)

\[L^{m}[I^{(2m)} u (x, t)] = k^{2m} u (x, t).\]  

(17)

Now

\[I^{(2m)} u (x, t) = k^{2m} \int_{\Omega} W^{2m} (x, t; \xi, \tau) u (\xi, \tau) \, d\xi \, d\tau\]

\[= \frac{k^{2m}}{2^{2m} \Gamma (m)} \int_{\Omega} J_{0} (ks) u (\xi, \tau) \, d\xi \, ds.\]
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so that (17) gives

\[
\frac{1}{2^{2m-1}m^{-1}} \Gamma(m) \int_{D} J_0(ks) u(\xi, \tau) d\xi d\tau = u(x, t)
\]

which is our lemma, equations (9) and (10).

4. Solution of the Wave Equation (2)

To solve the wave equation (2) with the initial conditions

\[
u(x, t) \bigg|_{t=t_0} = f(x), \quad \frac{\partial u}{\partial t} \bigg|_{t=t_0} = g(x),
\]

we consider the integral

\[
G = \int_{D} (u A v - v A u) d\xi d\tau
\]

with

\[
v = (ks) J_1(ks).
\]

Putting \(a = m\) in (12) and noting that

\[
v = 2^{2m} \pi^{-1} \Gamma(m + 1) W(2m+1),
\]

we get

\[
A v = 2mk^2 J_0(ks)
\]

Using (22) and (2) in (19),

\[
G = \int_{D} u A v d\xi d\tau
\]

\[
= 2mk^2 \int_{D} J_0(ks) u(\xi, \tau) d\xi d\tau
\]

\[
= 2mk^2 U(x, t)
\]

[from (9)]

On the other hand, we can transform the integral (19) into a surface integral by using Green's theorem. If \((\omega, \lambda_1, \lambda_2, \ldots, \lambda_{2m-1})\) denote the direction cosines of the outward drawn normal to the boundary, \(S\), of \(D\), we get

\[
G = \int_{S} \left[ \left( u \frac{\partial v}{\partial \tau} - v \frac{\partial u}{\partial \tau} \right) \omega + \sum_{i=1}^{2m-1} \left( u \frac{\partial v}{\partial \xi_i} - v \frac{\partial u}{\partial \xi_i} \right) \lambda_i \right] dV.
\]
The surface, $S$, is made up of two parts, $V$ and $V'$ (see Fig. 1). On $V$,

$$\tau = t_0, \quad r^2 \leq (t - t_0)^2$$

$$\omega = 1, \quad \lambda_1 = \lambda_2 = \ldots = \lambda_{2m-1} = 0.$$ 

Hence, the integral over $V$ is

$$G_V = \int_V \left[ u \frac{\partial v}{\partial \tau} - v \frac{\partial u}{\partial \tau} \right] dV$$

$$= \int_V \left[ f(\xi) \frac{\partial v}{\partial \xi} - g(\xi) v \right] dV$$

[using (18)]

$$= \int_V \left[ f(\xi) \cdot \frac{\partial v}{\partial t} - g(\xi) v \right] dV$$

$$= -\frac{\partial}{\partial t} \int_V f(\xi) v dV - \int_V g(\xi) v dV$$

On the surface $V'$ (Fig. 1), $s^2 = 0$, so that $v = 0$. Therefore, from (24), the integral over $V'$ becomes

$$G_{V'} = \int_{V'} \left[ u \frac{\partial v}{\partial \tau} \omega - \sum_{i=1}^{2m-1} u \frac{\partial v}{\partial \xi_i} \lambda_i \right] dV$$

We have

$$\omega = \frac{1}{\sqrt{2}}, \quad \lambda_i = \frac{\xi_i - x_i}{\sqrt{2}(t_0 - \tau)}$$

Now

$$\frac{\partial v}{\partial \tau} = \frac{\partial}{\partial \tau} \frac{(ks)}{\partial \xi} \left[ (ks) J_1 (ks) \right]$$

$$= k \frac{\partial}{\partial \tau} (ks) J_0 (ks)$$

and

$$\lambda_i \frac{\partial v}{\partial \xi_i} = \lambda_i \frac{\partial}{\partial \xi_i} \left[ (ks) J_1 (ks) \right]$$

$$= k \lambda_i \frac{\partial}{\partial \xi_i} (ks) J_0 (ks)$$
Both (28a) and (28b) are proportional to $s$ and therefore vanish on the surface $V'$. It follows that

$$G_{v'} = 0 \quad (29)$$

so that (24) becomes [see (25) and (29)]

$$G = G_v + G_{v'} = G_v \quad (30)$$

From (23) and (30), we get

$$2mk^2 U(x, t) = -\frac{\partial}{\partial t} \int_V f(\xi) \, v \, dV - \int_V g(\xi) \, v \, dV \quad (31)$$

Operating on both sides by $L^m$ and using (10), we now obtain

$$2^{2m}n^{m-1} \Gamma(m+1) k^2 u(x, t) = -L^m \left[ \frac{\partial}{\partial t} \int_V f(\xi) \, v \, dV + \int_V g(\xi) \, v \, dV \right]$$

Hence, the solution of the wave equation (2) with the initial conditions (18) is

$$u(x, t) = -\frac{1}{2^{2m}n^{m-1} \Gamma(m+1) k^2} \cdot L^m \left[ \frac{\partial}{\partial t} \int_V f(\xi) \, v \, dV \right.$$

$$+ \int_V g(\xi) \, v \, dV \left. \right] \quad (32)$$

where $L$ is the operator defined in (3), the function $v$ is defined by (20) and $V$ is the hyperplane $\tau = t_0$ bounding the retrograde characteristic cone (Fig. 1).

**SUMMARY**

A method given by Copson has been generalised here to obtain a solution of Cauchy's problem for the wave equation $\partial^2 u/\partial t^2 - \nabla^2 u + k^2 u = 0$ in any odd number of spatial dimensions. The method does not involve the use of any device for evaluating the divergent integrals.

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REFERENCES