A NOTE ON THE GENERALISATION OF HERMITE POLYNOMIALS

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1. Recently P. C. Chatterjee (1955) and Kurt Endl (1955 and 1956) have given a generalisation of the Hermite polynomials, independently, in the following manner:

Chatterjee

Even:

$$H_{km}(z) = \frac{(-1)^m k^{2m} \Gamma\left(m + \frac{1}{k}\right)}{\Gamma\left(\frac{1}{k}\right)} \, {}_1F_1\left(-m; \frac{1}{k}; z^k\right)$$

Odd:

$$H_{km+1}(z) = \frac{(-1)^m k^{2m+1} z \, \Gamma\left(m + \frac{1}{k} + 1\right)}{\Gamma\left(\frac{1}{k} + 1\right)} \times {}_1F_1\left(-m; \frac{1}{k} + 1; z^k\right). \tag{1}$$

Endl:

$$P_{n,k}(z) = z^{n_k^*} L_{n_k}^{\left(\frac{2n_k^* + 1}{k} - 1\right)}(z^k) \tag{2}$$

with

$$n = n_k^* + k n_k,$$

where

$$0 \leqslant n_k^* < k, \quad n_k^* \text{ and } k \text{ are integers.}$$

The associated Laguerre polynomials are defined by

$$L_n^{\alpha}(x) = \frac{e^x x^{-\alpha}}{n!}\left(\frac{d}{dx}\right)^n (e^{-x}x^{n+\alpha}) = \frac{(n+\alpha)!}{n!\,\alpha!}\,{}_1F_1(-n; \alpha+1; x) \quad (3)$$

In all these ${}_1F_1$ stands for the confluent hypergeometric function. It is easy to see that we can express (1) in terms of the associated Laguerre polynomials (3); for

$$\left.\begin{array}{l} \text{Even:} \\ \quad H_{km}(z) = (-1)^m k^{2m} m!\, L_m^{(1/k-1)}(z^k) \\ \text{Odd:} \\ \quad H_{km+1}(z) = (-1)^m k^{2m+1} m!\, L_m^{(1/k)}(z^k) \end{array}\right\} \quad (4)$$

Thus the generalisation by Chatterjee is nothing but putting the associated Laguerre polynomials in a different form. This fact has also been observed by A. Erdélyi (1956). Comparing (2) with (4) we note that the generalisation by Endl is also of the same nature.

E. T. Bell (1934) generalised the Hermite polynomials in the following form:

$$\xi_n(x, t, r) = e^{xt^r}\left(\frac{\partial}{\partial t}\right)^n \left(e^{-xt^r}\right) \quad (5)$$

It appears that Bell's work remained unnoticed by the later workers. In the present note we study some new properties of these polynomials.

2. Bell gave the following generating function for $\xi_n(x, t, r)$:

$$e^{x\,[t^r - (h+t)^r]} = \sum_{n=0}^{\infty} \frac{h^m}{n!}\,\xi_n(x, t; r) \quad (6)$$

Squaring (6) and rearranging the forms we get the generalisation of the well-known product theorem for the classical case:

$$2^{m/r}\xi_m(x, 2^{1/r}t, r) = \sum_{j=0}^{m} \binom{m}{j} \xi_j(x, t, r)\,\xi_{m-j}(x, t, r) \quad (7)$$

3. A determinantal representation:

We shall use the following lemma of Dave Pandres (Jr.) (1957):

Lemma.—If $Y = e^U$, where U is a function of t of class C^n, then $d^nY/dt^n = |\Delta_n|\,Y$, where

$$|\Delta_n| = \begin{vmatrix} D_1 & -1 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ D_2 & D_1 & -2 & 0 & 0 & \cdot & \cdot & 0 \\ D_3 & D_2 & D_1 & -3 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ D_n & D_{n-1} & \cdot & \cdot & \cdot & \cdot & D_2 & D_1 \end{vmatrix}$$

and

$$D_j = \frac{1}{(j-1)!} \frac{\partial^j U}{\partial t^j}$$

In this putting $U = -xt^r$, and noting that

$$\frac{\partial^j}{\partial t^j} t^r = 0 \qquad \text{if } j > r$$

$$= \frac{r!}{(r-j)!} t^{r-j} \qquad \text{if } j \leqslant r,$$

we get

$$\xi_n(x, t, r) = |\Delta_n|, \tag{8}$$

with

$$D_j = -jx \binom{r}{j} t^{r-j}$$

Hence the order of $\xi_n(x, t, r)$ is $n(r-1)$.

When we put $r = 2$, $x = 1$, we fall back to the classical case. This gives a new determinantal representation for $H_n(x)$ different from that given by Nielsen (1926) (I am thankful to Mr. T. S. Chihara of Seattle University, for pointing out this reference to me), namely

$$H_n(t) = \begin{vmatrix} 2t & 1 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 2 & 2t & 2 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 2 & 2t & 3 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2t \end{vmatrix} \tag{9}$$

4. In the following we shall give the generalisation of the known representation of the classical Hermite polynomials, namely

$$H_n(t) = (D - 2t)^n \cdot 1,$$

where $(D - 2t)^n$ is to be taken in the iterative sense. The derivation of this is based on the following lemma which can be easily proved by the method of induction:

Lemma.—If U, V are functions of t of class C^n then

$$D^n (e^V V) = e^U (D + U')^n V \tag{10}$$

where $D \equiv (\partial/\partial t)$; $U' = \partial U/\partial t$; and $(D + U')^n$ is to be taken in the iterative sense.

In (10) putting $U = -xt^r$, $V = 1$, we get the desired expression: (11)

$$\xi_n(x, t, r) = (D - xrt^{r-1})^n \cdot 1. \tag{12}$$

5. Truesdell's (1948, 1950) F-equation is

$$\frac{\partial}{\partial t} F(t, \alpha) = F(t, \alpha + 1). \tag{13}$$

Bell (1934) gave the recurrence relation:

$$\xi_{n+1}(x, t, r) = -rxt^{r-1} \xi_n(x, t, r) + \frac{\partial}{\partial t} \xi_n(x, t, r) \tag{14}$$

which is nothing but

$$e^{-xt^r} \xi_{n+1}(x, t, r) = \frac{\partial}{\partial t} (e^{-xt^r} \xi_n(x, t, r).$$

Hence

$$F(t, n) = e^{-xt^r} \xi_n(x, t, r) \tag{15}$$

satisfies (13).

(*a*) Using the following lemma due to Truesdell, namely:

"If $F(t, \alpha)$ satisfies (13) we have

$$F(t, \alpha + n) = \frac{\partial^n}{\partial t^n} F(t, \alpha)" \tag{16}$$

we get in the present case

$$\xi_{n+a}(x, t, r) = e^{xt^r} D^n [e^{-xt^r} \xi_a(x, t, r)] \tag{17}$$

Using (10), we get

$$\xi_{n+a}(x, t, r) = (D - rxt^{r-1})^n \cdot \xi_a(x, t, r) \tag{18}$$

(*b*) Now using the lemma by Truesdell, namely:

"If $F(t, a)$ is an analytic solution of the F-equation,

then

$$F(t + h, a) = \sum_{n=0}^{\infty} \frac{h^n}{n!} F(t, a + n)\text{"} \tag{19}$$

we get

$$e^{x[t^r - (t+h)^r]} \xi_a(x, t + h, r) = \sum_{n=0}^{\infty} \frac{h^n}{n!} \xi_{a+n}(x, t, r) \tag{20}$$

which generalises (6).

(*c*) Truesdell (1950) established the multiplication theorem for $F(t, a)$:

"If $F(t, a)$ satisfies the F-equation, then

$$F(kt, a) = \sum_{n=0}^{\infty} \frac{(k-1)^n t^n}{n!} F(t, a + n) \tag{22}$$

where k is a scalar and is such as to make the right-hand side of (22) convergent."

This gives us the multiplication formula for $\xi_n(x, t, r)$ as

$$e^{xt^r(1 - k^r)} \xi_a(x, kt, r) = \sum_{n=0}^{\infty} \frac{(k-1)^n t^n}{n!} \xi_{a+n}(x, t, r) \tag{23}$$

Noting that $\xi_0 = 1$, we have from (23) an interesting relation:

$$e^{xt^r(1 - k^r)} = \sum_{n=0}^{\infty} \frac{(k-1)^n t^n}{n!} \xi_n(x, t, r) \tag{24}$$

for all k for which (24) is convergent.

(*d*) From the integral for $F(t, a)$ given by Truesdell (1948):

" If $F(t, \alpha)$ is a solution of the F-equation and if it vanishes for $t = 0$ and $t = t_0$, then there exists an $f(\beta)$ such that

$$\int_0^{t_0} t^{\alpha+\beta} F(t, \alpha)\, dt = e^{i\alpha\pi} \Gamma(\alpha + \beta + 1) f(\beta), \tag{25}$$

$$Re(\alpha + \beta) > -1.$$

Putting $t_0 = \infty$, in this, and using (14) for $F(t, \alpha)$, we have

$$\int_0^{\infty} e^{-xt^r} t^{\alpha+\beta} \xi_\alpha(x, t, r)\, dt = f(\beta)\, e^{i\alpha\pi} \Gamma(\alpha + \beta + 1)$$

Let $\alpha = 0$, then

$$\int_0^{\infty} e^{-xt^r} t^{\beta} dt = f(\beta)\, \Gamma(\beta + 1) = \frac{1}{r} \frac{\Gamma\left(\frac{\beta+1}{r}\right)}{x^{\frac{\beta+1}{r}}}\,.$$

Hence

$$\int_0^{\infty} e^{-xt^r} t^{\beta+\alpha} \xi_\alpha(x, t, r)\, dt = \frac{e^{i\alpha\pi} \Gamma(\alpha + \beta + 1)\, \Gamma\left(\frac{\beta+1}{r}\right)}{r x^{\frac{\beta+1}{r}}\, \Gamma(\beta + 1)} \tag{26}$$

provided

$$Re(\alpha + \beta) > -1.$$

The classical case has been given by Truesdell (1948).

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