THE CHARACTERISTIC ROOTS OF THE PRODUCT OF TWO MATRICES

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1. INTRODUCTION AND NOTATIONS

Let A and B be two n-square matrices with elements in the field of complex numbers. In a recent paper this author has proved certain results regarding the upper bounds of the characteristic roots, \( c(AB) \), of the product matrix AB (i) in terms of the characteristic roots of A and B when both A and B are Hermitian and (ii) in terms of the characteristic roots of A, \((B + B^*)/2 \) and \((B - B^*)/2i \) when A is Hermitian and B is any matrix.

In this paper, we find the upper and lower limits for the real and imaginary parts of \( c(AB) \) in terms of the characteristic roots of the associated Hermitian matrices \((A + A^*)/2, (A - A^*)/2i, (B + B^*)/2 \) and \((B - B^*)/2i \), when A and B are any complex matrices.

In what follows, let us set \( H_1 = (A + A^*)/2, \ K_1 = (A - A^*)/2i, \ H_2 = (B + B^*)/2\) and \( K_2 = (B - B^*)/2i.\)

2. THE BOUND THEOREMS OF CHARACTERISTIC ROOTS

THEOREM 1.—Let A and B be two commuting n-square complex matrices, such that the Hermitian matrices \((A + A^*)/2, (A - A^*)/2i, (B + B^*)/2\) and \((B - B^*)/2i\) are at least positive semi-definite. Then

\[
\begin{align*}
&c_m\left(\frac{A + A^*}{2}\right) c_m\left(\frac{B + B^*}{2}\right) - c_m\left(\frac{A - A^*}{2i}\right) c_m\left(\frac{B - B^*}{2i}\right) \\
&\leq \text{Re} \ c(AB) \leq c_m\left(\frac{A + A^*}{2i}\right) c_m\left(\frac{B + B^*}{2i}\right) \\
&- c_m\left(\frac{A - A^*}{2i}\right) c_m\left(\frac{B - B^*}{2i}\right).
\end{align*}
\]

(1)
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and

\[ c_m \left( \frac{A + A^*}{2} \right) c_m \left( \frac{B - B^*}{2i} \right) + c_m \left( \frac{A - A^*}{2i} \right) c_m \left( \frac{B + B^*}{2} \right) \]

\[ \leq \operatorname{Im} c(AB) \leq c_M \left( \frac{A + A^*}{2} \right) c_M \left( \frac{B - B^*}{2i} \right) \]

\[ + c_M \left( \frac{A - A^*}{2i} \right) c_M \left( \frac{B + B^*}{2} \right), \]

(2)

where \( c_M \) and \( c_m \) stand respectively for the greatest and least characteristic roots.

Proof.—Every square matrix \( A = (A + A^*)/2 + (A - A^*)/2 = H_1 + iK_1 \), where \( H_1 \) and \( K_1 \) are Hermitian matrices. Let the characteristic roots (necessarily real) of \( H_1 \) be \( c_m(H_1) = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n = c_M(H_1) \).

Then there exists a unitary matrix \( P \) such that

\[ P^*H_1P = \text{diag.}(\lambda_1, \lambda_2, \ldots, \lambda_n) = D(\lambda), \]

whence

\[ H_1 = PD(\lambda)^*P. \] (3)

Since the matrix \( K_1 \) is also Hermitian, there exists a unitary matrix \( Q \), such that

\[ Q^*K_1Q = \text{diag.}(\gamma_1, \gamma_2, \ldots, \gamma_n) = D(\gamma), \]

where \( c_m(K_1) = \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n = c_M(K_1) \), are the (real) characteristic roots of \( K_1 \).

Therefore,

\[ K_1 = QD(\gamma)^*Q. \] (4)

Equations (3) and (4) give

\[ A = PD(\gamma)^*P^* + iQD(\gamma)^*Q^*. \] (5)

Similarly, let \( c_m(H_2) = \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n = c_M(H_2) \) be the (real) characteristic roots of \( H_2 \), \( c_m(K_2) = \nu_1 \leq \nu_2 \leq \ldots \leq \nu_n = c_M(K_2) \) be the roots of \( K_2 \), so that there exist unitary matrices \( R \) and \( S \) such that

\[ B = RD(\mu)^*R^* + iSD(\nu)^*S^* \] (6)

where \( D(\mu) \equiv \text{diag.}(\mu_1, \mu_2, \ldots, \mu_n) \) and \( D(\nu) \equiv \text{diag.}(\nu_1, \nu_2, \ldots, \nu_n) \).

From (5) and (6) we have

\[ AB = PD(\lambda)^*P^*RD(\mu)^*R^* - QD(\gamma)^*QSD(\nu)^*S^* + i[PD(\lambda)^*P^*SD(\nu)^*S^* + QD(\gamma)^*RD(\mu)^*R^*]. \] (7)
and

\[ A^*B^* = \text{PD} (\lambda) \text{P}^* \text{RD} (\mu) \text{R}^* - \text{QD} (\nu) \text{Q}^* \text{SD} (\nu) \text{S}^* \]
\[ - i [\text{PD} (\lambda) \text{P}^* \text{SD} (\nu) \text{S}^* + \text{QD} (\nu) \text{Q}^* \text{RD} (\mu) \text{R}^*]. \] (8)

Now if \( \sigma \) is a characteristic root of \( AB \), there exists a non-zero column vector \( x = (x_1, x_2, \ldots, x_n) \), with complex co-ordinates, such that

\[ \sigma x = ABx \]
or,

\[ \sigma x^* = x^* ABx, \] (9)

Taking the conjugate transpose of both the sides of (9), we have

\[ \sigma x^* = x^* (B^*A^*) x = x^* A^* B^* x, \] (10)
since \( AB = BA \) implies that \( B^*A^* = A^*B^* \).

Adding (9) and (10), we have

\[ (\sigma + \delta) x^* x = x^* [AB + A^*B^*] x \]
or,

\[ \left( \frac{\sigma + \delta}{2} \right) x^* x = x^* [\text{PD} (\lambda) \text{P}^* \text{RD} (\mu) \text{R}^*] x \]
\[ - x^* [\text{QD} (\nu) \text{Q}^* \text{SD} (\nu) \text{S}^*] x. \] (11)

By hypothesis, the Hermitian matrices \( H_1, K_1, H_2, K_2 \) are at least positive semi-definite (to be called p.s.d.), so that \( \lambda_i \geq 0, \gamma_1 \geq 0, \mu_i \geq 0, \nu_1 \geq 0 \) for \( i = 1, 2, \ldots, n \). Therefore the value of the first Hermitian form on the right-hand side of (11) at least does not decrease if we replace the diagonal matrices \( D (\lambda) \) and \( D (\mu) \) respectively by the scalar matrices \( \lambda_n I \) and \( \mu_n I \), and the value of the second Hermitian form at least does not increase if we replace the diagonal matrices \( D (\nu) \) and \( D (\nu) \) respectively by the scalar matrices \( \gamma_1 I \) and \( \nu_1 I \). Thus we have

\[ \left( \frac{\sigma + \delta}{2} \right) x^* x \leq \lambda_n \mu_n (x^* PP^* RR^* x) - \gamma_1 \nu_1 (x^* QQ^* SS^* x) \]
\[ = (\lambda_n \mu_n - \gamma_1 \nu_1) x^* x, \]
the matrices \( P, Q, R \) and \( S \) being unitary.

Since \( x^* x > 0 \), we obtain

\[ \left( \frac{\sigma + \delta}{2} \right) \leq \lambda_n \mu_n - \gamma_1 \nu_1, \]
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or,

\[ \Re \alpha (AB) \leq c_m (H_1) c_m (H_2) - c_m (K_1) c_m (K_2). \]  

(12)

In order to find the lower bound for \((a + e)/2\) we observe that the value of the first Hermitian form on the right-hand side of (11) does not increase if we replace the diagonal matrices \(D (\lambda)\) and \(D (\mu)\) respectively by the scalar matrices \(\lambda I\) and \(\mu I\), and the value of the second Hermitian form on the right-hand side of (11) does not decrease if we replace the diagonal matrices \(D (\gamma)\) and \(D (\nu)\) respectively by the scalar matrices \(\gamma I\) and \(\nu I\). Thus we have

\[ \frac{\alpha + \overline{\alpha}}{2} \geq \lambda \mu_1 (x^* P P^* R R^* x) - \gamma \nu_n (x^* Q Q^* S S^* x) \]

\[ = (\lambda \mu_1 - \gamma \nu_n) x^* x. \]

Since \(x^* x > 0\), we have

\[ \left( \frac{\alpha + \overline{\alpha}}{2} \right) \geq \lambda \mu_1 - \gamma \nu_n \]

or,

\[ \Re \alpha (AB) \geq c_m (H_1) c_m (H_2) - c_m (K_1) c_m (K_2). \]  

(13)

Combining (12) and (13), we obtain (1).

In order to prove (2), we subtract (10) from (9), and have

\[ (\sigma - \overline{\sigma}) x^* x = x^* [AB - A^* B^*] x, \]

\[ = 2ix^* [PD (\lambda) P^* SD (\nu) S^*] x + 2ix^* [QD (\gamma) Q^* RD (\mu) R^*] x \]

or,

\[ \left( \frac{\sigma - \overline{\sigma}}{2i} \right) x^* x = x^* [PD (\lambda) P^* SD (\nu) S^* + QD (\gamma) Q^* RD (\mu) R^*] x. \]  

(14)

By hypothesis, \(\lambda_i, \gamma_i, \mu_i\) and \(\nu_i\) for \(i = 1, 2, \ldots, n\) are non-negative. Therefore the value of the Hermitian form on the right-hand side of (14) does not decrease if we replace the diagonal matrices \(D (\lambda), D (\nu), D (\gamma)\) and \(D (\mu)\) respectively by the scalar matrices \(\lambda I, \nu I, \gamma I\) and \(\mu I\), and we have

\[ \frac{\sigma - \overline{\sigma}}{2i} \leq \lambda \nu_n + \gamma \mu_n, \text{ since } x^* x > 0, \]

or,

\[ \Im \alpha (AB) \leq c_m (H_1) c_m (H_2) + c_m (K_1) c_m (K_2). \]  

(15)
Similarly, we can prove that
\[ \text{Im} \ c(AB) \geq c_m(H_1) c_m(K_2) + c_m(K_1) c_m(H_2). \]  

(16)

Combining (15) and (16) we establish (2) and this completes the proof of the theorem.

If the matrices A and B of Theorem 1 are such that the Hermitian matrices \( H_1, K_1, H_2, K_2 \) are at least negative semi-definite (to be called n.s.d.), their characteristic roots are \( \leq 0 \), so that in this case the above theorem takes the following form:

**Theorem 2.**—Let \( A \) and \( B \) be two commuting \( n \)-square complex matrices, such that the Hermitian matrices \( (A - A^*)/2, \ (A - A^*)/2i, \ (B + B^*)/2 \) and \( (B - B^*)/2i \) are at least negative semi-definite. Then

\[
\begin{align*}
&\leq \text{Re} \ c(AB) \leq c_m \left( \frac{A + A^*}{2} \right) c_m \left( \frac{B + B^*}{2} \right) \\
&- c_m \left( \frac{A - A^*}{2i} \right) c_m \left( \frac{B - B^*}{2} \right),
\end{align*}
\]

(17)

and

\[
\begin{align*}
&\leq \text{Im} \ c(AB) \leq c_m \left( \frac{A + A^*}{2} \right) c_m \left( \frac{B - B^*}{2} \right) \\
&+ c_m \left( \frac{A - A^*}{2} \right) c_m \left( \frac{B + B^*}{2} \right),
\end{align*}
\]

(18)

where \( c_m \) and \( c_m \) denote respectively the greatest and least characteristic roots.

We omit the proof of this theorem and consider the following:

**Theorem 3.**—Let \( A \) and \( B \) be two commuting \( n \)-square complex matrices such that \((A + A^*)/2\) and \((A - A^*)/2i\) are at least p.s.d., while \((B + B^*)/2\) and \((B - B^*)/2i\) are at least negative semi-definite. Then
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\[
c_m \left( \frac{A + A^*}{2} \right) c_m \left( \frac{B + B^*}{2} \right) - c_m \left( \frac{A - A^*}{2i} \right) c_m \left( \frac{B - B^*}{2i} \right)
\leq \text{Re} \, c(AB) \leq c_m \left( \frac{A - A^*}{2} \right) c_m \left( \frac{B + B^*}{2} \right)
\]

\[
- c_m \left( \frac{A - A^*}{2i} \right) c_m \left( \frac{B - B^*}{2i} \right),
\]

(19)

and

\[
c_m \left( \frac{A + A^*}{2} \right) c_m \left( \frac{B - B^*}{2i} \right) + c_m \left( \frac{A - A^*}{2i} \right) c_m \left( \frac{B + B^*}{2} \right)
\leq \text{Im} \, c(AB) \leq c_m \left( \frac{A + A^*}{2} \right) c_m \left( \frac{B - B^*}{2i} \right)
\]

\[
+ c_m \left( \frac{A - A^*}{2i} \right) c_m \left( \frac{B + B^*}{2} \right),
\]

(20)

where \( c_m \) and \( c_m \) denote respectively the greatest and least characteristic roots.

**Proof.**—Since the Hermitian matrices \( H_1 \) and \( K_1 \) are at least p.s.d., \( \lambda_i \geq 0 \) and \( \gamma_i \geq 0 \) for \( i = 1, 2, \ldots, n \); and since \( H_2 \) and \( K_2 \) are at least n.s.d., \( \mu_i \leq 0 \) and \( \nu_i \leq 0 \) for \( i = 1, 2, \ldots, n \). Now, if we replace the diagonal matrices \( D(\lambda), D(\mu), D(\gamma) \) and \( D(\nu) \) in (11) respectively by the scalar matrices \( \lambda_1 I, \mu_1 I, \gamma_1 I \) and \( \nu_1 I \) the value of the Hermitian form on the right-hand side does not decrease. Hence, we have

\[
\sigma + \frac{\hat{\sigma}}{2} \leq \lambda_1 \mu_1 - \gamma_1 \nu_1,
\]

or,

\[
\text{Re} \, c(AB) \leq c_m (H_1) c_m (H_2) - c_m (K_1) c_m (K_2). \tag{21}
\]

Similarly, replacing the diagonal matrices \( D(\lambda), D(\mu), D(\gamma) \) and \( D(\nu) \) in (11) by \( \lambda_n I, \mu_n I, \gamma_n I \) and \( \nu_n I \) we can prove that

\[
\sigma + \frac{\hat{\sigma}}{2} \geq \lambda_n \mu_n - \gamma_n \nu_n,
\]

or,

\[
\text{Re} \, c(AB) \geq c_m (H_1) c_m (H_2) - c_m (K_1) c_m (K_2). \tag{22}
\]
Combining the inequalities (21) and (22), we establish (19).

Remembering the facts about the signs of the characteristic roots of the Hermitian matrices stated in the beginning of the proof of this theorem and replacing the diagonal matrices in (14) by suitable scalar matrices, we can likewise establish (20), and this completes the proof of Theorem 3.

So far we have considered only those matrices A and B for which the two pairs of the associated Hermitian matrices \((A + A^*)/2, (A - A^*)/2i\) and \((B + B^*)/2, (B - B^*)2i\) are p.s.d. or n.s.d., or one pair is p.s.d. the other being n.s.d. We now consider the following theorem in which one pair of the Hermitian matrices is indefinite, while the other is at least p.s.d.:

**Theorem 4.**—Let A and B be two commuting n-square complex matrices, such that \((A + A^*)/2, (A - A^*)/2i\) are indefinite, and \((B + B^*)/2, (B - B^*)/2i\) are at least positive semi-definite. Then

\[
c_m\left(\frac{A + A^*}{2}\right) c_M\left(\frac{B + B^*}{2}\right) - c_M\left(\frac{A - A^*}{2i}\right) c_M\left(\frac{B - B^*}{2i}\right)
\leq \text{Re} \; c(AB) \leq c_m\left(\frac{A + A^*}{2}\right) c_M\left(\frac{B + B^*}{2}\right)
- c_m\left(\frac{A - A^*}{2i}\right) c_M\left(\frac{B - B^*}{2i}\right),
\]

and

\[
c_m\left(\frac{A + A^*}{2}\right) c_M\left(\frac{B - B^*}{2i}\right) + c_m\left(\frac{A - A^*}{2i}\right) c_M\left(\frac{B + B^*}{2}\right)
\leq \text{Im} \; c(AB) \leq c_m\left(\frac{A + A^*}{2}\right) c_M\left(\frac{B - B^*}{2i}\right)
+ c_m\left(\frac{A - A^*}{2i}\right) c_M\left(\frac{B + B^*}{2}\right),
\]

where \(c_m\) and \(c_M\) denote respectively the greatest and least characteristic roots.

The inequalities of this theorem can be established by replacing the diagonal matrices of (11) and (14) by suitable scalar matrices taking into account the facts that since \(H_1\) and \(K_1\) are indefinite, \(\lambda_n > 0, \lambda_1 < 0, \gamma_n > 0\) and \(\gamma_1 < 0\) and since \(H_2\) and \(K_2\) are at least p.s.d., their characteristic roots are non-negative.
Similar results may be proved when the Hermitian matrices $H_1$ and $K_2$ are at least p.s.d. and

either

(i) one of the two Hermitian matrices $H_1$ and $K_2$ is indefinite, the other being at least n.s.d.;

or

(ii) one of the Hermitian matrices $H_1$ and $K_2$ is at least n.s.d., the other being at least p.s.d.;

or

(iii) one of the Hermitian matrices $H_1$ and $K_2$ is at least n.s.d., the other being indefinite.

3. SOME SPECIAL CASES OF THE ABOVE THEOREMS

We now give certain examples to show that many interesting results can be obtained as particular cases of the above theorems.

(i) If we put $B = I$ in the above theorems, we have

$$c_m\left(\frac{A + A^*}{2}\right) \leq \text{Re} \ c(A) \leq c_M\left(\frac{A + A^*}{2}\right), \quad (26)$$

and

$$c_m\left(\frac{A - A^*}{2i}\right) \leq \text{Im} \ c(A) \leq c_M\left(\frac{A - A^*}{2i}\right), \quad (27)$$

results due to Hirsch and Bromwich.

(ii) If the two Hermitian matrices $A$ and $B$ commute and are at least p.s.d., then $AB$ is also Hermitian so that all $c(AB)$ are real and non-negative. The limits for $c(AB)$ will, then, be given by

$$c_m(A) \ c_m(B) \leq c(AB) \leq c_M(A) \ c_M(B).$$

(iii) If $A$ is skew-Hermitian, then $A + A^* = 0$. If, moreover, $A$ is such that the Hermitian matrix $(A - A^*)/2i = -iA$ is at least p.s.d., then every $c(-iA)$ is real and $\geq 0$, so that $c(iA) \leq 0$. Also, $c_m(-iA) = -c_m(iA)$, $c_m(-iA) = -c_m(iA)$. Similar relations hold for $B$ if it is skew-Hermitian and $(B - B^*)/2i$ is at least p.s.d., so that $c(-AB)$ are real.
and $\geq 0$ and $c(AB) \leq 0$. For the commuting skew-Hermitian matrices $A$ and $B$ defined above, (1) reduces to

$$- c_m (-iA) c_m (-iB) \leq c(AB) \leq - c_m (-iA) c_m (-iB)$$

or,

$$- c_m (iA) c_m (iB) \leq c(AB) \leq - c_m (iA) c_m (iB)$$

or,

$$c_m (iA) c_m (iB) \leq c(-AB) \leq c_m (iA) c_m (iB).$$

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REFERENCES

