# THE ENUMERATION OF POSITIVE RATIONAL NUMBERS

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## §1. Introduction

The aim of this paper is to indicate a method of enumeration of positive rational numbers. Such an enumeration consists in establishing a (1 — 1) correspondence between the elements of the set $R^+$ of positive rationals (or a subset of it) and those of the set $I^+$ of positive integers. A few methods of enumeration of rationals have been indicated in Wilder[4] (pp. 80–81, 106–107).

In the following, we shall also study some of the properties of the median series (defined below) which is akin to the Farey series and also about the $\tau$-function defined by Hermes.[1] We make use of the properties of the median series and the $\tau$-function to enumerate the positive rationals $> 1$.

Hermes, on the basis of his $\tau$-function, proves certain results which, we are bound to think, are motivated by the structure of the median series studied in the paper. However Hermes does not obtain all the results obtained here and in particular Hermes does not interpret the binary scale representation of an integer $n$ in relation to the structure of the median series.

Hermes' paper was as a matter of fact suggested by a paper of Stern[2] in which he introduces the series called the Stern's series

$$S_0 \ldots (r, s)$$
$$S_1 \ldots (r, r+s, s)$$
$$S_2 \ldots (r, 2r+s, r+s, r+2s, s)$$

and so on. If $r = 0$, $s = 1$ in $S_i$ then this series becomes identical with the numerator of the median series $M_i$. If $r = 1$, $s = 0$, the series becomes identical with the denominator of $M_i$. From our point of view the importance of Stern's series is subordinate to that of the median series.

The significance of the partial quotients in the division transformation for two numbers $a$ and $b$ relatively prime, is also brought out in terms of the median approximations defined in Section 5. These approximations are also the approximations of Vahlen[3] but they differ from those of Vahlen in the method of introduction. While Vahlen introduces the approximations to a real number $\rho$ in connection with Farey series and is able to prove the important result that the approximations can be arranged in a single series $a_1, a_2, a_3, \ldots\ldots$ so that $a_i, a_{i+1}$ are the two consecutive terms of $F_i$, the Farey series of $i$-th order, between which the real number $\rho$ lies, we however define the same approximations in relation to the structure of the median series and Vahlen's series arrangement is no longer true for the median approximations defined by us. The relation between the Farey series and median series do not seem to be fully known though we can show that the Farey series $F_i$ is a part of the median series $M_i$ but not a part of $M_{i-1}$. The latter part is evident since the terms $1/i$ belong to $M_i$ but not to $M_{i-1}$. To prove that $F_i \subset M_i$ we can use the following result which can be easily proved: "The sum of the partial quotients of a rational number $m/n$ in lowest terms is equal to or less than the greater of $m$ and $n$."

Hermes' $\tau$-function is introduced here in Section 6 specifically by means of the order in $M_i$ and the recurrence relations on which Hermes bases his definition of the $\tau$-function are directly deduced in Section 6, from the order properties of the median series.

The results in Sections 5 and 6 are believed to be new. The auxiliary properties of medians, necessary for proving them, have been collected in Section 4.

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## §2. Notations and Definitions

2.1. The $\tau$-function of Hermes is defined by the recurrence relation

$$\tau_1 = 1 \text{ and } \tau_n = \tau_{(n-2^{\nu})} + \tau_{(2^{\nu+1}+1-n)}$$

where $2^{\nu} < n \leq 2^{\nu+1}$.

2.2. The Gaussian bracket is defined by (2.1.1)

$$[\ \ ] = 1, \qquad [a] = a \text{ and}$$

$$[a, \beta, \ldots, \gamma, \delta, \epsilon, \eta, \ldots, \zeta] \tag{2.2.1}$$

$$= [a, \beta, \ldots, \delta]\,[\epsilon, \eta, \ldots \zeta] + [a, \beta, \ldots, \gamma]\,[\eta, \ldots, \zeta].$$

2.3. If any positive integer $n$ has the unique alternating binary scale expansion

$$n = 2^{a_0} - 2^{a_1} + 2^{a_2} - \ldots\ldots + 2^{a_\lambda}, \tag{2.3.1}$$

$(0 \leqslant a_0 < a_1 < a_2 < \ldots . < a_\lambda)$ with odd number of terms (*i.e.*, $\lambda$ even), then

$$\tau_n = \tau\,(a_0, a_1 - a_0, a_2 - a_1, \ldots, a_\lambda - a_{\lambda-1}) \tag{2.3.2}$$

$$= (1 + a_0, a_1 - a_0, a_2 - a_1, \ldots, a_\lambda - a_{\lambda-1}) \tag{2.3.3}$$

[*vide* Hermes[1] (satz 1)]. The fact that the alternating binary scale expansion is unique for a given integer and that there are an odd number of terms in such an expansion is true for $n = 1, 2, 3$ for, $1 = 2^\circ$, $2 = 2^1$, $3 = 2^\circ - 2^1 + 2^2$. Let us now assume that this is true for all integers less than N (say) where $2^{k-1} < N \leq 2^k$. Now N can be uniquely written in the form

$$N = 2^{k-1} + 2^{a_1} + 2^{a_2} + \ldots . + 2^{a_n}$$

where $k - 1 > a_1 > a_2 > \ldots . > a_n \geq 0.$ (2.3.4)

Let now the alternating binary scale expansion of $N - 2^{k-1}$ be

$$N - 2^{k-1} = 2^{a_1+1} - 2^b + \ldots . \qquad \text{(odd number of terms)}$$

Then

$$N = 2^{k-1} + 2^{a_1+1} - 2^b + \ldots . \qquad \text{(even number of terms)}$$

$$= 2.2^{k-1} - 2^{k-1} + 2^{a_1+1} - 2^b + \ldots .$$

$$= 2^k - 2^{k-1} + 2^{a_1+1} - 2^b + \ldots . \qquad \text{(odd number of terms).}$$

This is an alternating binary scale representation of N with an odd number of terms and this is unique because (2.3.4) is unique.

2.4. The Median series is defined as follows:

$$M_0 = \left(\frac{0}{1}, \frac{1}{0}\right)$$

$$M_1 = \left(\frac{0}{1}, \frac{1}{1}, \frac{1}{0}\right)$$

$$M_2 = \left(\frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0}\right)$$

$$M_3 = \left(\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0}\right)$$

and so on.

## §3. Division Transformation and Euclidean Algorithm

It is known that given any two positive integers $a$ and $b$, an Euclidean algorithm can be constructed in the form

$$\begin{aligned}
a &= ba_0 + b_0 && (0 \leq b_0 < b)\\
b &= b_0a_1 + b_1 && (0 \leq b_1 < b_0)\\
b_0 &= b_1a_2 + b_2 && (0 \leq b_2 < b_1)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
b_{n-3} &= b_{n-2}\, a_{n-1} + b_{n-1} && (0 \leq b_{n-1} < b_{n-2})\\
b_{n-2} &= b_{n-1}\, a_n && (a_n > 1) \qquad (3.1)
\end{aligned}$$

which ultimately yields the Greatest Common Divisor (hereafter denoted as (GCD) of $a$ and $b$. The numbers $a_0$, $a_1$, $a_2$, ...., $a_n$ which appear in the successive stages of the division transformation are called the partial quotients in the Euclidean algorithm. It is also known that the rational $a/b$ can be put in the form of a regular finite continued fraction (hereafter denoted as CF)

$$\frac{a}{b} = c_0 + \frac{1}{c_1} + \frac{1}{c_2} + \cdots\cdots + \frac{1}{c_n}. \qquad (3.2)$$

The numbers $c_0$, $c_1$, $c_2$, .... are called the partial quotients of the continued fraction. It is easily seen that the equations

$$a_i = c_i \qquad (i = 0, 1, 2, 3, \ldots, n)$$

are true; that is, the partial quotients obtained in the above two operations are respectively the same.

The number of partial quotients can always be made odd; for if it is even, then $a_n$ can be replaced by the two partial quotients $(a_{n-1}, 1)$ so that the CF becomes

$$\frac{a}{b} = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} + \frac{1}{1}.$$

In (3.1) $b_{n-1}$ is the GCD of $a$ and $b$. The remainders $b_0$, $b_1$, $b_2$, ...., $b_{n-2}$ at each stage of the operation can be eliminated in terms of $a_0$, $a_1$, $a_2$, ...., $a_n$ and $b_{n-1}$ and then we get

$$\begin{aligned}
b_{n-2} &= b_{n-1}\, a_n\\
b_{n-3} &= b_{n-1}\, [a_{n-1}, a_n]\\
b_{n-4} &= b_{n-1}\, [a_{n-2}, a_{n-1}, a_n]\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
b_0 &= b_{n-1}\, [a_2, a_3, \ldots, a_n].
\end{aligned}$$

Finally we have

$$b = b_{n-1}\,[a_1, a_2, a_3, \ldots, a_n]$$

and

$$a = b_{n-1}\,(a_0, a_1, a_2, \ldots, a_n) \tag{3.3}$$

Since $b_{n-1}$ is the GCD of $a$ and $b$, it follows that

the integers represented by the two Gaussian brackets in (3.3) are relatively prime. (3.4)

The result of the elimination is also the expression of $a/b$ as the regular finite continued fraction

$$\frac{a}{b} = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$$

$$= \frac{[a_0, a_1, a_2, \ldots, a_n]}{[a_1, a_2, a_3, \ldots, a_n]}.$$

It follows that if $a/b$ is in its lowest terms

$$a = [a_0, a_1, a_2, \ldots, a_n]$$

$$b = [a_1, a_2, a_3, \ldots, a_n]. \tag{3.5}$$

From the above we see that the positive rational $R = a/b$ can be made to correspond to a sequence of integers $a_0, a_1, \ldots, a_n$ ($a_0 \geq 0$, $a_i > 0$ for $i = 1, 2, 3, \ldots, n$) which are the partial quotients in the CF expansion of R and these partial quotients completely specify the rational number. If $a/b > 1$ (so that $a_0 \geq 1$) is in its lowest terms, then the reverse of the CF is equal to another rational number

$$R' = \frac{[a_n, a_{n-1}, \ldots, a_1, a_0]}{[a_{n-1}, a_{n-2}, \ldots, a_1, a_0]}$$

$$= \frac{p_n}{p_{n-1}}$$

where $p_n$ is the numerator of the $n$-th convergent of the original CF. We can call R′ as the associate of R. It is clear that the relation between a rational number and its associate is symmetrical.

Therefore we obtain that if $a_i \geq 1$ ($i = 0, 1, 2, \ldots, n$)

$$[a_0, a_1, \ldots, a_n] = [a_n, a_{n-1}, \ldots, a_1, a_0]. \tag{3.6}$$

With any rational number $R = a/b$ we shall associate the function $\sigma(R) =$ the sum of the partial quotients in the CF development of R; *i.e.*,

$\sigma(R) = a_0 + a_1 + \ldots. + a_n$. If R and R′ are associates, then $\sigma(R) = \sigma(R')$. The significance of this function is brought out later on.

## §4. The Median and the Elementary Properties of the Median Series

We define the median of two positive rationals $a/b$ and $c/d$ (in lowest terms) to be $(a + c)/(b + d)$. Some of the important properties of the median are

4.1.1. The median $m_1/m_2$ of two unequal positive rationals $a/b$ and $c/d$ lies between them and is different from them (a property of ratios).

4.1.2. The determinants $bc - ad$, $m_1b - m_2a$, $m_2c - m_1d$ of $a/b$, $m_1/m_2$, $c/d$ (taken two by two) are equal. Conversely, if the determinants of the above type are equal for three ratios, then each one is the median of the other two.

4.1.3. If $bc - ad = 1$, then the median of $a/b$ and $c/d$ is in lowest terms.

4.1.4. If $m_1/m_2$ is the median of $a/b$ and $c/d$, then $m_2/m_1$ is the median of $b/a$ and $d/c$.

For the construction of the median series, we start with the extreme irreducible fractions 0/1 and 1/0. We denote $M_0 = (0/1, 1/0)$. Any $M_{i+1}$, $i = 0, 1, 2, \ldots.$ is obtained from $M_i$ by itroducing between every two successive terms of $M_i$ their median. Thus we get the median series mentioned in (2.4). $M_i$ is called the median series of $i$-th order. The following are some of the basic properties of the median series:

4.2.1. Every term in each of the median series is in lowest terms and the terms in a median series are in natural order from left to right and are all distinct. Further any two successive terms in a median series have determinant 1.

4.2.2. Every term of $M_i$ not in $M_{i-1}$ is the median of the two adjacent terms in $M_{i-1}$. These terms are irreducible. Every term of $M_i$ which occurs also in $M_{i-1}$ is the median of the two adjacent terms in $M_i$. These terms are however necessarily reducible.

4.2.3. The median series $M_i$ contains $2^i + 1$ terms. Hence the even (odd) terms as counted from the left are also the even (odd) terms as counted from the right. The terms in $M_{i+1}$ that are not in $M_i$ are the even terms and these continue as odd terms in succeeding series.

4.2.4. The $r$-th term from the left in $M_i$ is the reciprocal of the $r$-th term from the right. So the middle term is its own reciprocal and so is 1. (It can be seen that the median series of $i$-th order is got by adding to the Farey series of $i$-th order, its reflection about 1.)

4.2.5. The middle term of the left half of $M_i$ is $1/2$. The $r$-th terms to the right and the left of $1/2$ add up to 1.

4.2.6. The $r$-th term from the left (right) in $M_i$ occurs as the $(2r-1)$-th term from the left (right) in $M_{i+1}$; hence as $(4r-3)$-th term in $M_{i+2}$ and so in general as $\{2^k(r-1)+1\}$-th term in $M_{i+k}$.

4.2.7. So if $2^c$ is the highest power of 2 dividing $r-1$, then $r$-th term from right (left) of $M_i$ occurs in $M_{i-c}$ as the $2\alpha$-th term from the right (left) where

$$\alpha = \frac{r-1}{2^{c+1}} + \tfrac{1}{2}.$$

Being an even term, this term occurs for the first time in $M_{i-c}$.

## §5. THE MEDIAN SERIES, (VAHLEN'S) APPROXIMATIONS AND THE ENUMERATION OF POSITIVE RATIONAL NUMBERS

The following is an interpretation based on the idea of approximations to a rational number in terms of the partial quotients $a_0, a_1, a_2, \ldots, a_n$ of the CF expansion of the rational number and this brings out the relation between the approximations and the median series and also the fact that the rational number $\rho$ occurs for the first time (and so as an even term) in $M_{\sigma(\rho)}$.

It is clear that $M_0$ has no term betewen $\rho$ and $1/0$. It can be seen that the largest value of $i$ such that there is no term between $\rho$ and $1/0$ in $M_i$ is $a_0$. The penultimate elements $1, 2, 3, \ldots, a_0$ in $M_1, M_2, \ldots, M_{a_0}$ respectively, may therefore be called the first $a_0$ median approximations to $\rho$.*

Again the largest value of $i$ such that in $M_{a_0+i}$ there is no term between $\rho$ and $a_0/1$ is evidently $a_1$. The numbers $a_0+1/1, a_0+1/2, \ldots, a_0+1/a_1$ of $M_{a_0+1}, M_{a_0+2}, \ldots, M_{a_0+a_1}$ respectively, in the interval $(\rho, 1/0)$ give the second set of median approximations to $\rho$.† Proceeding thus we have

5.1. $a_i$ is the maximum value of $t$ such that there is no term in $M_{a_0+a_1+a_2+\cdots+a_{i-1}+t}$ between $\rho$ and $a_0+1/a_1+1/a_2+\ldots+1/a_{i-1}$ $(i=2, 3, \ldots, n)$.

* The successive terms in $M_i$ between which $\rho$ lies are $(i/1, 1/0)$, $i=1, 2, 3, \ldots, a_0$.

† The successive terms in $M_{a_0+i}$ between which $\rho$ lies are $(a_0/1, a_0+1/i)$, $i=1, 2, 3, \ldots, a_1$.

5.2. The $i$th set of approximations to $\rho$ are: $a_0 + 1/a_1 + 1/a_2 + \ldots + 1/a_{i-2} + 1/t$ $(t = 1, 2, \ldots, a_{i-1};\ i = 2, 3, \ldots, n+1)$.

5.3. Thus the total number of approximations to $\rho$ is $a_0 + a_1 + \ldots + a_n = \sigma(\rho)$. This shows that $\rho$ occurs for the first time in $M_{\sigma(\rho)}$. It is to be noted that all the convergents of the CF occur among these approximations.

We now proceed to find the exact order in which the convergents $a_0$, $a_0 + 1/a_1$, $a_0 + 1/a_1 + 1/a_2, \ldots$ occur in the respective median series $M_{a_0}$, $M_{a_0+a_1}, \ldots$. From the above reasoning relating to the approximations it is clear that

5.4. $a_0$ and $1/0$ are consecutive in $M_{a_0}$ and $\rho$ lies between them;

5.5. $a_0$ and $a_0 + 1/a_1$ are consecutive in $M_{a_0+a_1}$ and $\rho$ lies between them; and so on. We note that $a_0$ occurs in $M_{a_0}$ as the $2^{a_0}$-th term and hence occurs in $M_{a_0+a_1}$ as $\{2^{a_1}(2^{a_0} - 1) + 1\}$-th term. Since $a_0 + 1/a_1$ is the term next to $a_0$ in $M_{a_0+a_1}$ its order is $2^{a_1}(2^{a_0} - 1) + 2$. Hence $a_0 + 1/a_1$ occurs in $M_{a_0+a_1+a_2}$ as the $\{2^{a_2}(2^{a_1+a_0} - 2^{a_1} + 1) + 1\}$-th term. Since $a_0 + 1/a_1 + 1/a_2$ is the term preceding $a_0 + 1/a_1$ in $M_{a_0+a_1+a_2}$ its order is $2^{a_0+a_1a_2} - 2^{a_1+a_2} + 2^{a_2}$. Hence in general

5.6. the $i$-th convergent $a_0 + 1/a_1 + 1/a_2 + \ldots + 1/a$ occurs in $M_{a_0+a_1+\cdots+a_i}$ as the $p$-th term where $p = 2^{a_0+a_1+\cdots+a_i} - 2^{a_1+\cdots+a_i} + \ldots + (-1)^i 2^{a_i} + \{1 + (-1)^{i+1}\}$.

*Proof.*—Let this be true for the $i$-th convergent. We shall prove that this is then true when $i$ is replaced by $i + 1$. The term $(-1)^{i+1}$ indicates that if $i$ is old, the order of $(i+1)$-th convergent in $M_{a_0+\cdots+a_{i+1}}$ is one more and if $i$ is even one less, than the order in $M_{a_0+\cdots a_{i+1}}$ of the $i$-th convergent. From (5.6) and (4.2.6) the $i$-th convergent has in $M_{a_0+\cdots+a_{i+1}}$, the order

$$2^{a_{i+1}}(p-1) + 1 = 2^{a_0+\cdots+a_{i+1}} - \ldots + (-1)^{i+1} 2^{a_{i+1}} + 1.$$

Hence the $(i+1)$-th convergent has in $M_{a_0+\cdots+a_{i+1}}$ the order equal to

$$2^{a_0+\cdots+a_{i+1}} - \ldots + (-1)^{i+1} 2^{a_{i+1}} + \{1 + (-1)^{i+2}\},$$

which is the same as (5.6) with $(i+1)$ instead of $i$. Since the result has been verified for $i = 1, 2$ it holds generally.

The above is also true for the $n$-th convergent, that is $\rho$. For the sake of convenience let us take $n$ to be even so that the CF has an odd number of terms. We then have the

THEOREM 1. *A rational number* $\rho$ *occurs for the first time as* q*-th term in* $M_{\sigma(\rho)}$; *if* $\rho$ *has partial quotients* $a_0, a_1, \ldots, a_n$ (n *even*) *then,*

$$q = 2^{a_0+\cdots+a_n} - 2^{a_1+\cdots+a_n} + \ldots + 2^{a_n}$$

$$\rho = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}.$$

It can be seen that since $a_n > 0$, $q$ is even as ought to be the case since $\rho$ occurs in $M_{\sigma(\rho)}$ for the first time.

COROLLARY.—The number $\rho$ occurs as the $2r$-th term of $M_{\sigma(\rho)}$ where

$$\sigma = \sigma(\rho) = a_0 + a_1 + \ldots + a_n \qquad (n \text{ even})$$

$$r = 2^{\sigma-1} - 2^{\sigma-1-a_0} + \ldots + 2^{a_n-1}.$$

*Note.*—If $\rho < 1$, then $a_0 = 0$ and in this case the first two terms of $r$ cancel each other.

From the corollary we see that we can associate with the partial quotients of $\rho$, the integer $r = 2^{\sigma-1} - 2^{\sigma-1-a_0} + \ldots + 2^{a_n-1}$, $\sigma = \sigma(\rho)$ and then we are in a position to say that $\rho$ occurs as the $2r$-th term in $M_{\sigma(\rho)}$. Conversely if we pick out any term in $M_i$ which occurs for the first time, its order is an even number $2r$ and we may show as follows that any such $r$ can be represented as an alternating binary scale representation with odd number of terms starting with $2^{\sigma-1}$.

For, we know that any integer $r$ can be uniquely expressed as an alternating binary scale representation with odd number of terms in the form

$$r = 2^{a_0} - 2^{a_1} + \ldots + 2^{a_n} \qquad (n \text{ even})$$

$$(a_0 > a_1 > a_2 > \ldots > a_n \geq 0).$$

If $r \geq 2^{i-2}$ (so that $\rho > 1$) its alternating binary scale representation will begin with $2^{i-1}$ and we can, by the corollary, obtain its partial quotients and in particular the first partial quotient of the corresponding $2r$-th term in $M_i$ will be greater than zero. If $r < 2^{i-2}$ (so that $\rho < 1$), the alternating binary scale representation will begin with $2^m$ $(m \leq i-2)$; we shall then add to this representation the initial term so that

$$r = 2^{i-1} - 2^{i-1} + 2^m - \ldots\ldots$$

The partial quotients are now given in order $(0, i-1-m, \ldots)$.

5.7. Thus we are not only able to give the precise order $2r$ of the number $\rho$ and the median series $M_\sigma$ in which it occurs for the first time but conversely, given any term, say $2r$-th in an arbitrary $M_i$, we are able to obtain its partial quotients.

Consequently, the corollary establishes a $(1-1)$ correspondence between a given rational number $\rho > 1$ and an integer $r$ in the form of an alternating binary scale representation with odd number of terms, making it possible to enumerate the rational numbers greater than 1.

We next deduce from the corollary, the following theorem which shall be used in the next section.

THEOREM 2. *The sequence of numbers which are the R-th terms from the right in all the median series, form an increasing arithmetic progression with common difference* 1.

*Proof*

*Case* 1.—Let R be an even number $2r$. Let $\rho$ be the $2r$-th term from the right in $M_i$ and let it have the partial quotients $a_0, a_1, \ldots, a_n$ ($n$ even). Since $2r$-th term from the right is the $(2^i - 2r + 2)$-th term from the left, we have by Theorem 1, $2(2^{i-1} - r + 1) = 2^i - 2^{i-a_0} + \ldots + 2^{a_n}$ ($n$ even) so that

$$r = 2^{i-1-a_0} - 2^{i-1-a_0-a_1} + \ldots + - 2^{a_{n-1}} + 1. \qquad (5.8)$$

This is therefore the unique alternating binary scale representation for $r$.

If $\rho'$ be the $2r$-th term from the right in $M_{i+k}$ similarly we have,

$$r = 2^{i+k-1-b_0} - 2^{i+k-1-b_0-b_1} + \ldots - 2^{b_{m-1}} + 1$$

where $b_0, b_1, \ldots, b_m$ are the partial quotients in the CF expansion of $\rho'$. Comparing this with (5.8) we have

$$b_0 = a_0 + k; \quad b_i = a_i\ (i > 0); \quad m = n.$$

Thus the terms in $M_i$ and $M_{i+k}$ differ only by $k$, the difference in the orders of the median series.

*Case* 2.—Let now R be an odd number $= 2^c j + 1$ ($j$ odd). Then by (4.2.7) it follows that R-th term of $M_i$ occurs as $(j+1)$-th term from the right in $M_{i-c}$ and this is even. Similarly R-th term from the right in $M_{i+k}$ occurs as $(j+1)$-th term from the right in $M_{i+k-c}$. Since $j+1$ is even, the arguments of case (i) hold here and we have the theorem.

From (4.2.4) we now have

5.9. If the R-th term from the left of $M_i$ is $a/b$, then the R-th term from the left of $M_{i+k}$ is $(a/ka+b)$.

5.10. Hence, the numerators of the R-th terms from the left in $M_i$, $M_{i+1}$, ...., ($R \leq 2^{i+1}$) are the same; that is, the numerator is independent of $i$. Also, the denominators form an arithmetic progression with common difference equal to the corresponding numerator.

## §6. THE RELATION BETWEEN THE TERMS OF THE MEDIAN SERIES AND THE $\tau$-FUNCTION

The corollary under Theorem 1 and (5.7) show that the rational numbers are completely enumerated by the sequence of the median series. If any rational number $\rho$ occurs for the first time in $M_\sigma$ as $2r$-th term, then we call $\sigma$, $r$ as the median co-ordinates of $\rho$. From what has preceded, the partial quotients of $\rho$ can be found by the expression of $r$ in the alternating binary scale. $\rho$ itself can be expressed by Gaussian brackets in terms of the partial quotients.

But we can adopt a short method by expressing the numerator and the denominator of $\rho$ directly as a function of the median co-ordinates. We may define the numerator of the $2r$-th term from the left in $M_i$ to be $\mu(r, i)$. But from (5.10) it follows that $\mu(r, i)$ is independent of $i$; that is, $\mu(r, i) = \mu(r)$ say. This is in fact the $\tau$-function of Hermes defined in Section 2. The relation between these two functions is given by $\mu(r) = \tau_r$. Also the denominator of the $2r$-th term in $M_i$ is the numerator of its reciprocal, that is, the $(2^i - 2r - + 2)$-th term. Therefore the denominator of the $2r$-th term is

$$\mu(2^{i-1} - r + 1) = \tau_{(2^{i-1}-r+1)}$$

Thus

$$2r\text{-th term of } M_i = \frac{\tau_r}{\tau_{(2^{i-1}-r+1)}} \quad (r \leq 2^{i-1}), \tag{6.1}$$

and this is in lowest terms.

We now proceed to show a method of calculating $\tau_r$.

We have seen that given a positive integer $r$ which has the unique alternating binary scale representation

$$r = 2^{i-1} - 2^{i-1-a_0} + 2^{i-1-a_0-a_1} - \ldots + 2^{a_n-1} \; (n \text{ even})$$
$$(a_0 \geq 0, a_n \geq 1),$$

it follows from (2.3.1) and (2.3.2) that

$$\begin{aligned} \tau_r &= [a_n, a_{n-1}, \ldots, a_0] \\ &= [a_0, a_1, \ldots, a_n] \text{ by } (3.6) \\ &= [a_2, a_3, \ldots, a_n] \text{ if } a_0 = 0. \end{aligned}$$

From the alternating binary scale representations of the integers 1, 2, 3, 4, 5, *viz.*,

$$1 = 2^{\circ};\ 2 = 2^1;\ 3 = 2^2 - 2^1 + 2^0;\ 4 = 2^2;\ 5 = 2^3 - 2^2 + 2^0,$$

we get

$$\tau_1 = 1,\quad \tau_2 = 2,\quad \tau_3 = 3,\quad \tau_4 = 3,\quad \tau_5 = 4.$$

The recurrence relation for $\tau_n$ can be obtained from (5.9). For, from (6.1)

$2r$-th term from the left in

$$M_i = \frac{\tau_r}{\tau_{(2^{i-1}-r+1)}} \quad (r \leq 2^{i-1})$$

Hence

$2r$-th term in

$$M_{i+1} = \frac{\tau_r}{\tau_{(2^i-r+1)}}$$

$$= \frac{\tau_r}{\tau_r + \tau_{(2^{i-1}-r+1)}} \qquad \text{(by 5 9)}$$

As these have been remarked to be in lowest terms we get,

$$\tau_{(2^i-r+1)} = \tau_r + \tau_{(2^{i-1}-r+1)} \qquad (r \leq 2^{i-1})$$

Replacing $i$ by $(i+1)$ and the $r$ by $(2^i - a + 1)$ we get

$$\tau_{(2^i+a)} = \tau_a + \tau_{(2^i-a+1)} \qquad (a \leq 2^i) \qquad (6.2)$$

This is precisely the recurrence relation of Hermes[1] (p. 372).

We can now calculate any term (say $k$-th from the left) of any $M_i$. Let the term be denoted by T $(k, i)$.

*Case* 1.—Let $k$ be even $= 2r$.

Then

$$T(k, i) = \frac{\tau_r}{\tau_{(2^{i-1}-r+1)}}$$

by (6.1)

$$= \frac{\tau_{k/2}}{\tau_{\frac{1}{2}(2^i-k+2)}}$$

*Case* 2.—When $k$ is odd, $k+1$ and $k-1$ are both even. Since $k$-th term of $M_i$ is the median of the adjacent terms, we have

$$T(k, i) = \frac{\tau_{\frac{1}{2}(k-1)} + \tau_{\frac{1}{2}(k+1)}}{\tau_{\frac{1}{2}(2^i-k+3)} + \tau_{\frac{1}{2}(2^i-k+1)}} \qquad (6.3)$$

Or alternately, when $k$ is odd, write $k = 2^c j + 1$ ($j$ odd). Then this term occurs for the first time as $(j+1)$-th term in $M_{i-c}$, *i.e.*, as the $(j+1)/2$-th even term.

Hence $T(k, i) = T(j+1, i-c)$. Hence by case (1) we have

$$T(k, i) = \frac{\tau_{(j+1)/2}}{\tau_{\frac{1}{2}(2^{i-c}-j+1)}} \qquad \text{where } k = 2^c j + 1. \qquad (6.4)$$

It can be verified that (6.3) and (6.4) are the same, using $\tau$-function properties.

## REFERENCES

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