ON A FUNCTION OF RAMANUJAN

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Section 1

Let

\[ \phi(t) = \sum_{n=1}^{\infty} \frac{n^{n-2}}{n-1} t^{n-1} e^{-nt} \text{ for } t \geq 1 \]

denote Ramanujan function.\(^1\) It was conjectured by Ramanujan that

\[ (-1)^n \frac{d^n \phi(t)}{dt^n} \geq 0 \text{ for } t \geq 1, \ n = 1, 2, 3, \ldots \ldots \]

The conjecture was proved by Auluck and Chowla\(^3\) for \(n = 1, 2, 3,\) and by Shah and Sharma\(^4\) for \(n = 1, 2, 3, 4.\) In this note I prove the result

\[ (-1)^n \left[ \frac{d^n \phi(t)}{dt^n} \right]_{t=1+0} > 0 \text{ for } n = 1, 2, 3, \ldots \ldots \]

I indicate a method which gives an easy proof for small values of \(n.\) If \(n\) is larger than the method in para 5 proves the conjecture for \(t > n\) but the proof for \(t < n\) for large values of \(n\) becomes complicated.

Section 2

**Theorem I.**

\[ (1 - \omega) \omega_n + (\omega_{n-1} - \phi_{n-1}) = \sum_{r=1}^{n-1} n^r \omega_r \omega_{n-r} \]

or

\[ = \sum_{r=1}^{n-2} n^r \omega_r \omega_{n-r} + \frac{1}{2} n^2 \omega_{n/2}^2 \]

according as \(n\) is odd or even.

**Proof.** \(-\phi(t)\) and its derivatives are continuous functions for \(t > 1.\)

From

\[ \omega e^{-\omega} = te^{-t} \]

\[ (1 - \omega) \omega_1 = (1 - t) \phi \]
next
\[(1 - \omega) \omega_2 + (\omega_1 - \phi_1) = \omega_1^2 = \frac{1}{2} C_1 \omega_1^2\]
similarly
\[(1 - \omega) \omega_3 + (\omega_2 - \phi_2) = 3 \omega_1 \omega_2 = 3 C_1 \omega_1 \omega_2\]
\[(1 - \omega) \omega_4 + (\omega_3 - \phi_3) = 4 \omega_1 \omega_3 + 3 \omega_2^2 = 4 C_1 \omega_1 \omega_3 + \frac{1}{2} C_3 \omega_2^2\]
\[(1 - \omega) \omega_5 + (\omega_4 - \phi_4) = 5 C_1 \omega_1 \omega_4 + 5 C_2 \omega_2 \omega_3\]
\[(1 - \omega) \omega_6 + (\omega_5 - \phi_5) = 6 C_1 \omega_1 \omega_5 + 6 C_2 \omega_3 \omega_4 + \frac{1}{2} C_3 \omega_3^2\]

Suppose the relation is true for the \(n\)th value, \(i.e.,\)
\[(1 - \omega) \omega_n + (\omega_{n-1} - \phi_{n-1})\]
\[= \sum_{r=1}^{n-1} \frac{n}{2} C_r \omega_r \omega_{n-r} \quad \text{for } n \text{ odd}\]
\[= \sum_{r=1}^{n-2} \frac{n}{2} C_r \omega_r \omega_{n-r} + \frac{1}{2} n^n C_{n/2} \omega_{n/2}^2 \quad \text{for } n \text{ even}\]

by actual differentiation.
\[(1 - \omega) \omega_{n+1} + (\omega_n - \phi_n)\]
\[= (n C_0 + n C_1) \omega_1 \omega_n + (n C_1 + n C_2) \omega_2 \omega_{n-1} + \ldots + \left( \frac{n C_{n-2}}{2} + \frac{n C_{n-1}}{2} \right) \omega_{n-2} \omega_{n-3} + \frac{n C_{n-1}}{2} \omega_{n+1}^2\]

or
\[(1 - \omega) \omega_{n+1} + (\omega_n - \phi_n)\]
\[= \frac{n+1}{2} C_{n+1} \omega_1 \omega_n + \frac{n+1}{2} C_2 \omega_2 \omega_{n-1} + \ldots + \frac{n+1}{2} C_{n+1} \omega_{n-2} \omega_{n+2} + \frac{1}{2} \frac{n+1}{2} C_{n+1} \omega_{n+1}^2\]

or
\[(1 - \omega) \omega_{n+1} + (\omega_n - \phi_n)\]
\[= \sum_{r=1}^{(n+1)-2} \frac{n+1}{2} C_r \omega_r \omega_{(n+1)-r} + \frac{1}{2} \frac{n+1}{2} C_{n+1} \omega_{n+1}^2\]

here \((n + 1)\) is even and it satisfies condition (2) above. A similar procedure is followed for the even value of \(\lambda\). The theorem is thus proved from the principle of induction.
THEOREM II.

$$\omega_n = n\phi_{n-1} + \phi_nt$$

This follows by direct successive differentiation of

$$\omega = \phi t$$

Section 3

Let the values of $\phi$, $\phi_1$, $\phi_2$, etc., and $\omega$, $\omega_1$, $\omega_2$, etc., at $t = 1 + 0$ and $\omega = 1 - 0$ be respectively denoted by $\bar{\phi}$, $\bar{\phi}_1$, $\bar{\phi}_2$, etc., and $\bar{\omega}$, $\bar{\omega}_1$, $\bar{\omega}_2$, etc.

Now

$$\bar{\phi} = 1$$

So

$$\bar{\omega} = 1$$

Further from Theorems I and II $\bar{\omega}_1 = \bar{\phi} + \bar{\phi}_1 = 1 + \bar{\phi}_1$ and $\bar{\omega}_1 - \bar{\phi}_1 = \bar{\omega}_1^2$

giving $\bar{\phi}_1 = -2$ and $\bar{\omega}_1 = -1$.

Similarly,

$$\bar{\omega}_2 = 2\bar{\phi}_1 + \bar{\phi}_2 = -4 + \bar{\phi}_2$$

and $\bar{\omega}_2 - \bar{\phi} = 3\bar{\omega}_1\bar{\omega}_2$

So

$$\bar{\omega}_2 = \frac{4}{3} \text{ and } \bar{\phi}_2 = \frac{16}{3}$$

Next

$$\bar{\omega}_3 = 3\bar{\phi}_2 + \phi_3$$

and $\bar{\omega}_3 - \bar{\phi}_3 = 4\bar{\omega}_1\bar{\omega}_3 + 3\bar{\omega}_2^2$

So

$$\bar{\omega}_3 = -\frac{8}{3} \text{ and } \bar{\phi}_3 = -\frac{56}{3}$$

In this way

$$\phi_1 (1 + 0) = -2$$

$$\phi_2 (1 + 0) = \frac{16}{3}$$

$$\phi_3 (1 + 0) = -\frac{56}{3}$$
\[
\phi_4 (1 + 0) = \frac{3712}{45}
\]
\[
\phi_5 (1 + 0) = -\frac{11968}{27}.
\]

Thus a general method for determining any number of members of the sequence \(\phi_1 (1 + 0), \phi_2 (1 + 0), \phi_3 (1 + 0), \) etc., is developed. The direct differentiation method used by Auluck and Chowla and others becomes unworkable after \(n = 5\) as the differentiations are lengthy and huge and the connected inequalities are complicated.

**Section 4**

The sets of relations in Theorem I can be expressed in the form

\[
(1 - \omega) \, t \phi_2 + (1 - 2 \omega) \, \phi_1 + \omega_1 = \omega_1^2
\]
\[
(1 - \omega) \, t \phi_3 + (2 - 3 \omega) \, \phi_2 + \omega_2 = 3 \omega_1 \omega_2
\]
\[
(1 - \omega) \, t \phi_4 + (3 - 4 \omega) \, \phi_3 + \omega_3 = 4 \omega_1 \omega_3 + 3 \omega_2^2
\]
\[
(1 - \omega) \, t \phi_5 + (4 - 5 \omega) \, \phi_4 + \omega_4 = 5 \omega_1 \omega_4 + 10 \omega_2 \omega_3
\]
\[
\vdots
\]

\[
(1 - \omega) \, t \phi_n + (n - 1 - n \omega) \, \phi_{n-1} + \omega_{n-1}
\]
\[
= \sum_{r=1}^{\frac{n-1}{2}} nC_r \omega_r \omega_{n-r}
\]
\[
= \sum_{r=1}^{\frac{n-2}{2}} nC_r \omega_r \omega_{n-r} + \frac{1}{2} nC_{n/2} \omega_{n/2}^2
\]

as \(n\) is odd or even.

From Equation (1) Section (2)

\[
\omega_1 < 0 \text{ for } t > 1.
\]

Hence \(\bar{\omega}_1 < 0\) and so from Theorem II, \(\bar{\phi}_1 < 0\). Now from (ii) above

\[
\bar{\omega}_2 - \bar{\phi}_2 = 3\bar{\omega}_1 \bar{\omega}_2 \text{ using Theorem II, } 2\bar{\phi}_1 = 3\bar{\omega}_1 \bar{\omega}_2
\]

Hence

\[
\bar{\omega}_2 = \frac{2\bar{\phi}_1}{3\bar{\omega}_1} > 0 \text{ and so } \bar{\phi}_2 > 0.
\]
Next

\[ \bar{\omega}_3 - \bar{\phi}_3 = 4\bar{\omega}_1\bar{\omega}_2 + 3\bar{\omega}_2 \text{ or } 3\bar{\phi}_2 - 4\bar{\omega}_1\bar{\omega}_3 > 0 \]

:. \[ \bar{\omega}_3 < \frac{3\bar{\phi}_2}{4\bar{\omega}_1} < 0 \text{ and so } \bar{\phi}_3 < 0 \]

Similarly,

\[ 4\bar{\phi}_3 - 3\bar{\omega}_1\bar{\omega}_3 < 0 \text{ or } \bar{\omega}_4 > \frac{4\bar{\phi}_3}{5\bar{\omega}_1} > 0 \text{ and so } \bar{\phi}_4 > 0 \]

Let \( n \) be odd, so that \( n\bar{\phi}_{n-1} - (n + 1)\bar{\omega}_1\bar{\omega}_n < 0 \), i.e., \( \bar{\omega}_n < 0 \) and so \( \bar{\phi}_n < 0 \).

Now for \((n + 1)\)th value we have \((n + 1)\phi_n - (n + 2)\omega_1\omega_{n+1} < 0\)

Hence

\[ \bar{\omega}_{n+1} > 0 \text{ and so } \bar{\phi}_{n+1} > 0 \]

Similarly for even value of \( n \). Hence from the principle of induction whether \( n \) is even or odd \((-1)^n \frac{d^n \phi}{dt^n} > 0 \).

Thus the conjecture is proved for \( t = 1 + 0 \) or

\[ (-1)^n \left[ \frac{d^n \phi(t)}{dt^n} \right]_{t=0} > 0 \text{ for } n = 1, 2, 3, \ldots \ldots \]

Section 5

**THEOREM III.**

\[ \phi_{n+1} = nC_0\phi\omega_{n+1} + nC_1\phi_1\omega_n + nC_2\phi_2\omega_{n-1} + \ldots + nC_n\phi_n\omega_1 - \phi_n. \]

**Proof.**—From \( \omega e^{-\bar{\omega}} = te^{-t} \) we have \( \phi e^{-\omega} = e^{-t} \)

hence

\[ \frac{\phi}{\phi} = \phi_1 = \phi - 1. \]

Differentiating successively,

\[ \phi_{n+1} = nC_0\phi\omega_{n+1} + nC_1\phi_1\omega_n + nC_2\phi_2\omega_{n-1} + \ldots + nC_n\phi_n\omega_1 - \phi_n. \]

Section 6

**THEOREM IV.**

If \( t \geq n \) where \( n \) is an integer, \((-1)^r \frac{d^r \phi}{dt^r} > 0 \) for \( r = 1, 2, \ldots, n \).
From Theorem II,

\[ \omega_n = t \phi_n + n \phi_{n-1} \]

now

\[ \phi_1 = \phi \omega_1 - \phi \text{ or } t \phi_1 = \omega_1 - \omega \]

or

\[ \omega_1 - \phi = \omega_1 - \omega \text{ or } (1 - \omega) \omega_1 = \phi - \omega = \phi (1 - t) \]

this gives \( \omega_1 < 0 \) hence from \( \omega_1 = t \phi_1 + \phi \) we get \( \phi_1 < 0 \)

next \( \phi_2 = \phi \omega_2 + \phi_1 \omega_1 - \phi_1 \) or \( \omega_2 - 2 \phi_1 = \omega_2 + t \omega_1 + \phi_1 \) or \( (1 - \omega) \omega_2 = t \phi_1 \omega_1 + (2 - t) \phi_1 \) if \( t \geq 2 \) then \( \omega_2 > 0 \) and hence from \( \omega_2 = t \phi_2 + 2 \phi_1 \) we have \( \phi_2 > 0 \) or \( (1 - 2)^2 \frac{d^2 \phi}{dt^2} > 0 \). Similarly \( (1 - \omega) \omega_2 = 2t \phi_1 \omega_2 + t \phi_2 \omega_1 + (3 - t) \phi_2 \).

If \( t \geq 3 \) then \( \omega_3 < 0 \) hence from \( \omega_3 = t \phi_3 + 3 \phi_2 \) we have

\[ \phi_3 < 0, \text{i.e.,} (1)^3 \frac{d^3 \phi}{dt^3} > 0. \]

Suppose the result is correct for the \( n \)th value, i.e.,

\[ (1 - \omega) \omega_n = \sum t^{n-1} \frac{d^{n-1} \phi}{dt^{n-1}} \omega_n + \ldots + (n - t) \phi_{n-1} \]

and for

\[ t \geq n, (1)^n \omega_n > 0 \text{ and } (1)^n \frac{d^n \phi}{dt^n} > 0 \]

now from this result and the usual steps

\[ (1 - \omega) \omega_{n+1} = \sum t^{n} \phi_1 \omega_n + t^n \phi_2 \omega_{n-1} + \ldots + \{(n + 1) - t\} \phi_n \]

and if \( t > n + 1, (1)^{n+1} \omega_{n+1} > 0 \) and so \( (1)^{n+1} \phi_{n+1} > 0 \) thus the theorem follows from the principle of induction.

This proves the conjecture \( (1)^r \frac{d^r \phi}{dt^r} > 0 \)

for all integral values of \( r \) less than \( n \) provided, \( t \geq n \).

REFERENCES
1. Srinivasa Ramanujan \( \text{Collected Papers, 1927.} \)
4. \text{Bombay University Journal, 1948, 1-4.} \)
5. \text{American Mathematical Monthly, 1956, 63 (6), 407-8.}