SOME APPLICATIONS OF BANACH FUNCTIONAL METHODS TO SUMMABILITY

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§ 1. INTRODUCTION

The matrix \( A = (a_{ik}), \ i, k = 1, 2, 3, \ldots \) of real or complex constants transforms any given sequence \( x = \{x_n\} \) of real or complex numbers into a sequence \( y = \{A_i(x)\} \) by the equations

\[
(T) \quad y_i = A_i(x) = \sum_{k=1}^{\infty} a_{ik}x_k \quad (i = 1, 2, 3, \ldots).
\]

The sequence \( x \) is said to be summable by the matrix \( A \) to the value \( l \) if all the sums \( y_i \) are finite and \( \lim_{i \to \infty} y_i = l \). We denote the transformation \( (T) \) by the matrix equation \( y = Ax \), where \( y \) and \( x \) are column vectors whose elements in the \( i \)-th row are \( y_i \) and \( x_i \) respectively. For the sake of convenience we shall use the same symbol to represent a sequence and the column vector corresponding to it.

Let \( (c_0), (c) \) and \( (m) \) denote as usual the spaces of null sequences, convergent sequences and bounded sequences respectively, with the usual norm. If \( (X) \) and \( (Y) \) are two sets of sequences then \( (X) \cap (Y) \) will denote their intersection, \( i.e., \) the set of sequences (points) common to both \( (X) \) and \( (Y) \); and we shall denote by \( \Gamma(X, Y) \) the class of all matrices \( A \) which have the property that \( x \in (X) \) implies \( A x \in (Y) \). We define further, for every complex number \( \rho \) that \( \Gamma(c, \rho c) \) is the class of matrices \( A \) such that \( \lim_{i \to \infty} y_i = l \) implies \( \lim_{i \to \infty} \sum_{k=1}^{\infty} a_{ik}x_k = \rho l \). Therefore in particular \( \Gamma(c, c) \) is the set of all \( K \)-matrices (conservative or convergence-preserving matrices) and \( \Gamma(c, 1c) \) is the set of all \( T \)-matrices (permanent or regular matrices).

The summability field of \( A \), \( i.e., \) the set of all sequences \( x = \{x_n\} \) such that \( Ax \in \Gamma(c, c) \) will be denoted by the symbol \( A \), and if \( (X) \) is a given set of sequences, we define \( R_{\Gamma(X)}(A) \) to be the "range of \( A \) over \( (X) \)", \( i.e., \) as the set of sequences \( y \) such that \( y = Ax \) where \( x \in \Gamma(X) \).
The object of this paper is to extend some results that Banach [(1), pp. 90-95] has proved for $\Gamma'(c, 1c)$ to other classes of matrices and to derive therefrom a number of other theorems. The immediate extensions of Banach’s results and the extensions obtained therefrom of some results due respectively to Hanai (3), Hill (4) and Ramanujan (7) are given in §2; in §3 we show that applications of these functional analysis methods and result yield a number of theorems due respectively to Copping (2), Wilansky (8) and the author (5), (6). The proofs as well as some of the results of §3 are believed to be also new.

§2

We begin by recalling that the necessary and sufficient conditions in order that the matrix $A$ should belong to (i) $\Gamma(m, m)$, (ii) $\Gamma(c, c)$, (iii) $\Gamma(c_0, c_0)$, (iv) $\Gamma(c, p_c)$ are, respectively,

(i) (a): $\|A\| = \sup_{1 \leq i < \infty} \sum_{k=1}^{\infty} |a_{ik}| < \infty$

(ii) (a) is satisfied,

(b): $\lim_{i \to \infty} \sum_{k=1}^{\infty} a_{ik} = \rho$ exists, and

(c): $\lim_{i \to \infty} a_{ik} = \delta_k$ exists for each $k = 1, 2, 3, \ldots$

(iii) (a) and (c) are true, with $\delta_k = 0$ for all $k = 1, 2, 3, \ldots$

(iv) (a), (b) and (c) are true, with $\delta_k = 0$ for all $k = 1, 2, 3, \ldots$

Theorem 1.11.—If the matrix $A \in \Gamma'(c, c)$ satisfies the condition

$$\chi(A) = \lim_{i \to \infty} \sum_{n=1}^{\infty} a_{in} - \sum_{n=1}^{\infty} \lim_{i \to \infty} a_{in} \equiv \rho - \sum_{n=1}^{\infty} \delta_n \neq 0$$

and if $y = \{y_i\}$ is a convergent sequence such that

$$\sum_{i=1}^{\infty} a_i y_i = 0$$

whenever

$$\sum_{i=1}^{\infty} |a_i| < \infty \text{ and } \sum_{i=1}^{\infty} a_{i1} a_{1n} = 0 \text{ for all } n = 1, 2, 3, \ldots$$

$^1$ Banach [(1), p. 91] has proved this theorem for the case $A \in \Gamma'(c, 1c)$ when (2.1) is automatically satisfied.
then there exists for every number \( \epsilon > 0 \) a convergent sequence \( x \) such that

\[ | A_i(x) - y_i | < \epsilon \text{ for all } i = 1, 2, 3, \ldots \]  

(2.4)

**Proof.**—The conclusion of the theorem means that \( y \in \overline{\mathcal{R}(c)}(A) \), where \( \overline{\mathcal{R}(c)}(A) \) is the closure of the set of A-transforms of convergent sequences. By a lemma of Banach [(1), p. 57] it is plainly enough to prove that every linear continuous functional\(^2\) vanishing over \( \overline{\mathcal{R}(c)}(A) \) will vanish at \( y \). Now, any linear continuous functional defined over \( c \) is of the form [(1), p. 66].

\[ f(s) = c_0 s_0 + \sum_{n=1}^{\infty} c_n s_n, \quad |c_0| + \sum_{n=1}^{\infty} |c_n| < \infty \]  

(2.5)

where \( s = \{s_n\} \) has \( s_0 \) for limit. Therefore any linear continuous functional over \( \mathcal{R}(c)(A) \) will be of the form

\[ f(v) = f(Au) = c_0 [\rho u_0 + \sum \delta_p (u_p - u_0)] + \sum_{n=1}^{\infty} c_n \sum_{p=1}^{\infty} a_{np} u_p \]  

(2.6)

where \( u = \{u_i\} \) and \( \lim_{i \to \infty} u_i = u_0 \). Using the condition (2.1) and (ii) (a), we may write

\[ f(v) = \sum_{p=1}^{\infty} u_p b_p + c_0 u_0 \chi(A) \]

where

\[ b_p = c_0 \delta_p + \sum_{n=1}^{\infty} c_n a_{np} \text{ for all } p = 1, 2, 3, \ldots \]  

(2.7)

Now if the functional \( f \) vanishes over \( \mathcal{R}(c)(A) \), then taking \( u^{(n)} = \{\delta_i^n\} \), where \( \delta_i^n \) has the value unity if \( i = n \) and is zero otherwise, for \( n = 1, 2, 3, \ldots \) successively, we see that

\[ b_p = 0 \text{ for all } p = 1, 2, 3, \ldots \]  

(2.8)

Therefore form (2.6) it follows that \( f(v) = c_0 u_0 \chi(A) = 0 \).

Since by hypothesis \( \chi(A) \neq 0 \), we see by taking \( u^{(0)} = \{\delta_i^0\} \) that \( c_0 = 0 \). The equations (2.7) and (2.8) now give

\[ \sum_{n=1}^{\infty} c_n a_{np} = 0 \text{ for all } p = 1, 2, 3, \ldots \]

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\(^2\) We prefer this terminology to Banach's "linear functional" in order to avoid any possible confusion.
and since by (2.5) $\sum_{n=1}^{\infty} |c_n| < \infty$ it follows from the hypothesis that $\sum_{n=1}^{\infty} c_n y_n = 0$

and therefore $f(y) = c_0 \lim_{i \to \infty} y_i + \sum_{n=1}^{\infty} c_n y_n = 0$. This proves the theorem.

A trivial modification of the proof of Theorem 1.1 gives the following:

**Theorem 1.2.**—If $\mathbf{A} \in \Gamma(c_0, c_0)$ and $y \in (c_0)$ are such that (2.3): $\sum_{i=1}^{\infty} \alpha_i \gamma_i = 0$ implies (2.2): $\sum_{i=1}^{\infty} \alpha_i y_i = 0$, then there exists for every number $\epsilon > 0$ a null sequence $x$ such that $|A_i(x) - y_i| < \epsilon$ for all $i = 1, 2, 3, \ldots$.

**Theorem 2.**—(Generalised converse of Theorem 1.1.)

Let $\mathbf{A} \in \Gamma(m, m)$ and $y = \{y_i\}$ be such that for every $\epsilon > 0$ there exists a sequence $x \in (m)$ satisfying the condition

$|A_i(x) - y_i| < \epsilon$ for all $i = 1, 2, 3, \ldots$

Then for any sequence $\{\alpha_i\}$ such that

$\sum_{i=1}^{\infty} |\alpha_i| < \infty$ and $\sum_{i=1}^{\infty} \alpha_i \gamma_i = 0$ for all $n = 1, 2, 3, \ldots$ (2.3)

we shall have

$\sum_{i=1}^{\infty} \alpha_i y_i = 0$.

**Proof.**—Let $\{\alpha_i\}$ be a sequence satisfying the condition (2.3). Then

$\alpha(s) = \sum_{i=1}^{\infty} \alpha_i s_i$ is a linear continuous functional defined over $(m)$ and $\|\alpha\| = \sum_{i=1}^{\infty} |\alpha_i|$. Also, $y \in (m)$ since $Ax = \{A_i(x)\} \in (m)$ and therefore we have:

$|\alpha(Ax - y)| = \sum_{i=1}^{\infty} |\alpha_i| (\sum_{n=1}^{\infty} \alpha_i x_n - y_i)|$

$\leq \|\alpha\| \cdot \|Ax - y\| < \|\alpha\| \cdot \epsilon$ (2.9)

and since $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |\alpha_i| \cdot |\gamma_i| \cdot |x_n| \leq \|\alpha\| \cdot \|A\| \cdot \|x\| < \infty$

we have

$\alpha(Ax) = \sum_{i=1}^{\infty} \alpha_i \sum_{n=1}^{\infty} \alpha_i x_n$

$= \sum_{n=1}^{\infty} x_n \sum_{i=1}^{\infty} \alpha_i \gamma_i$

$= 0$ by (2.3).

# Hanai (3) has proved this theorem for the case $\mathbf{A} \in \Gamma(c, 1c)$. The proof given here is simpler.
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Therefore (2.9) gives

\[ \left| \sum_{i=1}^{\infty} a_i y_i \right| \leq \left| a (Ax - y) \right| + \left| a (Ax) \right| < \| a \| \cdot \epsilon. \]

Since \( \epsilon > 0 \)

is as small as we please, we see that \( \sum_{i=1}^{\infty} a_i y_i = 0. \)

**Definition.**—If the matrix \( A \) is such that for every sequence \( \{a_i\} \) the condition (2.3) implies that \( a_i = 0 \) for all \( i = 1, 2, 3, \ldots \), then \( A \) is said to be of type \( M \).

**Theorem 3.1.**—Let \( A \in \Gamma(c, c) \) be of type \( M \) and let \( \chi(A) \neq 0. \) Then \( \mathcal{R}_{(c)}(A) \) is dense in \( (c) \).

The proof is immediate after Theorem 1.1.

As a consequence of Theorems 1.2 and 2 we have the

**Theorem 3.2.**—The matrix \( A \in \Gamma(c_0, c_0) \) is of type \( M \) if and only if \( \mathcal{R}_{(c_0)}(A) \) is dense in \( (c_0) \).

**Theorem 3.3.**—Let \( A \in \Gamma(m, m) \). Then \( \mathcal{R}_{(c)}(A) \) is dense in \( (c) \) if and only if the set of sequences (points) consisting of the columns of the matrix \( A \) and the sequence of row-sums \( \{\sum_{n=1}^{\infty} a_{in}\} \) is fundamental in \( (c) \).

**Proof.**—If we denote by \( \Delta \) the linear space generated by the set of sequences \( \{\delta_i^{(n)}\}, n = 0, 1, 2, \ldots \), where \( \delta_i^{(n)} = 1 \) or \( 0 \) according as \( i = n \) or \( i \neq n \) for \( n > 0 \) and \( \delta_i^{(0)} = 1 \) for all \( i \), then it is well known that \( \Delta \) is \( (c) \). Now, the linear space generated by the set mentioned in the theorem is the same as the set \( \mathcal{R}_{\Delta}(A) \). Since \( \Delta \) is \( (c) \) and since \( A \) defines a linear continuous operation over the Banach space \( (m) \) [in virtue of (1), Theorem 1, p. 54] and the relations \( l. u. b. \ | \sum_{1 \leq i < \infty} a_{in} x_n | \leq l. u. b. \ | \sum_{1 \leq i < \infty} a_{in} | \cdot l. u. b. \ | x_n | = \| A \| \cdot \| x \| \)

it follows that \( \mathcal{R}_{\Delta}(A) \supset \mathcal{R}_{(c)}(A) \). Therefore we have:

\[ \mathcal{R}_{\Delta}(A) = \mathcal{R}_{\Delta}(A) \supset \mathcal{R}_{(c)}(A). \]

\(^4\) Banach [(1), p. 93] has proved this for \( A \in \Gamma(c, 1c) \) when the condition \( \chi(A) \neq 0 \) is automatically satisfied.

\(^5\) It may be remarked that Ramanujan (7) has proved this theorem for \( A \in \Gamma(c, 1c) \) without however making use of the condition (ii) (b) of §2.
But $R_\Delta(A) \subset R_{(c)}(A)$ and hence we see that $\overline{R_\Delta(A)} = R_{(c)}(A)$. This proves the theorem.

By arguing in exactly the same manner we get the result: \textit{Let $A \in \Gamma'(m, m)$. Then $R_{(c)}(A)$ is dense in $(c_0)$ if and only if the columns of $A$ form a set fundamental in $(c_0)$.} Theorem 3.2 now yields the following:

**Theorem 3.4.**\textit{The matrix $A \in \Gamma'(c_0, c_0)$ is of type M if and only if its columns form a set fundamental in $(c_0)$}.

**Theorem 4.**\textit{(Generalised converse of Theorem 3.1.) If $A \in \Gamma'(m, m)$ and $R_{(m)}(A)$ is dense in $(c_0)$ then $A$ is of type M.} \[\text{PROOF. -- Since } R_{(m)}(A) \supseteq (c_0) \text{ we see that for every } y \in (c_0) \text{ and every number } \epsilon > 0 \text{ there exists a bounded sequence } x \text{ such that} \]
\[|A_i(x) - y_i| < \epsilon \text{ for all } i = 1, 2, 3, \ldots \]

In virtue of Theorem 2, the condition (2.3) will then imply that $\sum_{i=1}^{\infty} a_i y_i = 0$. Since this is true for every null sequence $y$, we see by considering the sequences $y^{(n)} = \{\delta_{i}^{(m)}\}$, $n = 1, 2, 3, \ldots$ that $a_i = 0$ for all $i = 1, 2, 3, \ldots$ and the theorem is proved.

**Corollary.**\textit{The matrix $A \in \Gamma'(c, c)$, $\chi(A) \neq 0$, is of type M if and only if $R_{(c)}(A)$ is dense in $(c)$.} \[\text{THEOREM 5.1. -- If } A \in \Gamma'(c_0, c_0), \text{ then for every bounded sequence } x_0 \text{ such that } Ax_0 \in (c_0) \text{ and for every number } \epsilon > 0 \text{ there exists a null sequence } x \text{ satisfying the condition} \]
\[|A_i(x) - A_i(x_0)| < \epsilon \text{ for all } i = 1, 2, 3, \ldots \]

In other words, \textit{if } $A \in \Gamma'(c_0, c_0)$ \textit{then } $R_{(c)}(A)$ \textit{is dense in } $R_{(c_0)(m)}(A)$, where $(A_0)$ denotes the set of sequences which are summable by $A$ to the value 0.

**PROOF. -- Let us write } y_i = A_i(x_0) \text{ for each } i = 1, 2, 3, \ldots \text{ and let } \{a_i\} \text{ be a sequence such that } \sum_{i=1}^{\infty} |a_i| < \infty \text{ and } \sum_{i=1}^{\infty} a_ia_{in} = 0 \text{ for all } n = 1, 2, 3, \ldots \text{ We have then} \]

\[\text{\textsuperscript{6} Hill (4) has proved this for reversible matrices in } \Gamma'(c, 1c) \text{ though he does not seem to use the reversibility.}\]
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\[ \sum_{i=1}^{\infty} a_i y_i = \sum_{i=1}^{\infty} a_i A_i(x_0) = \sum_{i=1}^{\infty} a_i \sum_{n=1}^{\infty} a_{in} s_n \]
\[ = \sum_{n=1}^{\infty} s_n \sum_{i=1}^{\infty} a_{in} = 0 \]

(the order of summation being justified by the fact that \( \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} a_i | \cdot | a_{in} | s_n | \leq \sum_{i=1}^{\infty} a_i \cdot \| A \| \cdot \| s \| < \infty \)).

The result follows now from Theorem 1.2.

**Theorem 5.2.**—If \( A \in \Gamma(c, c) \) and \( \chi(A) \neq 0 \) then \( R_{(c)}(A) \) is dense in \( \mathbb{R}(c, \chi(A)) \).

**Proof.**—Same as for Theorem 5.1, only at the end we use Theorem 1.1 instead of Theorem 1.2.

**Theorem 5.3.**—Let \( A \in \Gamma(m, m) \) and let there exist for every number \( \epsilon > 0 \) and for every given \( x_0 \in \chi(A)(m) \) a sequence \( x \) such that
\[ | A_i(x) - A_i(x_0) | < \epsilon \text{ for all } i = 1, 2, 3, \ldots \]
Then for any sequence satisfying (2.3) we shall have \( \sum_{i=1}^{\infty} a_i A_i(x_0) = 0 \).

**Proof.**—We have only to write \( y = \{y_i\} = \{A_i(x_0)\} \) and apply Theorem 2.

§ 3

In this Section we apply the results obtained above to reciprocals of summability matrices.

**Theorem 6.1.**—Let \( A \in \Gamma(c, c) \), \( \chi(A) \neq 0 \) and let \( 'A \in \Gamma(m, m) \) where \( 'A \cdot A = I \), the identity. Then \( (A)(m) = (c) \).

**Proof.**—We note that \( 'A \) sums every \( x \in \mathbb{R}_{(c)}(A) \) for \( 'A(Au) = u \) for all \( u \in (c) \), indeed for all \( u \in (m) \). Now, by Theorem 5.2, \( \mathbb{R}(c)(A) \) is dense in \( \mathbb{R}(A_{(m)})(m) \); also, since \( 'A \in \Gamma(m, m) \) we see that \( 'A \) defines a linear continuous operation over \( (m) \) by a well-known result [Banach (1), Théorème 1, p. 54]. Therefore it follows that \( 'A \) sums every \( x \in \mathbb{R}(A_{(m)})(m) \). That is, if \( y \in (A)(m) \) and \( x = Ay \) then \( 'Ax = 'A(Ay) = y \) belongs to \( (c) \). It follows that \( (A)(m) \subset (c) \) and since \( 'A \in \Gamma(c, c) \) the theorem is proved.

**Theorem 6.2.**—If \( A \in \Gamma(c, c) \) and \( \chi(A) = 0 \), then there is no matrix \( 'A \in \Gamma(m, m) \) such that \( 'A \cdot A = I \).

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7 Banach [(1), p. 93] has proved this for the special case \( A \in \Gamma(c, 1 c) \).
8 This result has recently been announced by Wilansky (8).
PROOF.—It is well known that under the hypothesis of the theorem there exists a bounded divergent sequence summable by the method A, i.e., \((A)(m) \neq (c)\). The required result now follows from Theorem 6.1.

**THEOREM 6.3.**—If \(A \varepsilon \Gamma '(c, c)\) and \('A \varepsilon \Gamma '(m, m), 'A \cdot A = I, then (A)(m) = (c)\).

**PROOF.**—This follows immediately from Theorems 6.1 and 6.2.

**THEOREM 6.4.**—If \(A \varepsilon \Gamma (c, c)\) and \(A^{-1} \varepsilon \Gamma (m, m)\) is such that \(A^{-1} \cdot A = A \cdot A^{-1} = I, then A^{-1} \varepsilon \Gamma (c, c)\).

**PROOF.**—Let \(x = \{x_n\}\) be any convergent sequence. Then \(A(A^{-1}x) = x \varepsilon (c)\). Theorem 6.3 gives now that \(A^{-1}x \varepsilon (c)\). Since this is true for all \(x \varepsilon (c)\), the theorem is proved.

**THEOREM 6.5.**—If \(A \varepsilon \Gamma (c_0, c_0)\) and \('A \varepsilon \Gamma (m, m), 'A \cdot A = I, then (A_0)(m) = (c_0)\).

The proof is the same as that for Theorem 6.1, only read \((c_0)\) wherever \((c)\) occurs and use Theorem 5.1 instead of Theorem 5.2.

**THEOREM 6.6.**—If \(A \varepsilon \Gamma (c_0, c_0)\) and \(A^{-1} \varepsilon \Gamma (m, m)\) is such that \(A^{-1} \cdot A = A \cdot A^{-1} = I, then A^{-1} \varepsilon \Gamma (c_0, c_0)\).

The proof follows from Theorem 6.5 just as Theorem 6.4 followed from Theorem 6.3.

**REFERENCES**

6. .. "Note on a theorem of Mazur and Orlicz in summability" (To appear).

* See Copping (2) and Parameswaran (6) for two other proofs of this theorem.
* See Parameswaran (5) for a different proof of this theorem.