QUITE a number of interesting relations have been noticed in recent years (cf. the reference at the end of this paper) concerning the ultraspherical (and other orthogonal) polynomials, in connection with the proof of Turán's inequality which had an extraordinary resonance in the recent literature on Special Functions. In the course of proving Turán's inequality for Legendre polynomials, viz.,

\[ \Delta_n(x) = [P_n(x)]^2 - P_{n+1}(x) P_{n-1}(x) \geq 0, \quad (n \geq 1, \ |x| \leq 1), \quad (1) \]

B. S. Madhava Rao and V. R. Thiruvenkatachar\(^a\) established the convexity property of \( \Delta_n(x) \) by showing that

\[ \frac{d^2}{dx^2} \Delta_n(x) = -\frac{2}{n(n+1)} \left[ \frac{d}{dx} P_n(x) \right]^2, \quad (n \geq 1). \quad (2) \]

V. R. Thiruvenkatachar and T. S. Nanjundiah\(^b\) employed the formula

\[ (1 - x^2) D_n^{(\lambda)}(x) = \kappa_{n, \lambda} \Delta_n^{(\lambda)}(x) \quad (3) \]

where

\[ \Delta_n^{(\lambda)}(x) = \left( \frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)} \right)^2 - \left( \frac{P_{n+1}^{(\lambda)}(x)}{P_{n+1}^{(\lambda)}(1)} \right) \left( \frac{P_{n-1}^{(\lambda)}(x)}{P_{n-1}^{(\lambda)}(1)} \right), \quad (4) \]

\[ D_n^{(\lambda)}(x) = \left( \frac{d}{dx} P_n^{(\lambda)}(x) \right)^2 - \frac{d}{dx} P_{n+1}^{(\lambda)}(x) \frac{d}{dx} P_{n-1}^{(\lambda)}(x), \quad (5) \]

and

\[ \kappa_{n, \lambda} = n(n + 2\lambda) [P_n^{(\lambda)}(1)]^2 \]

for proving Turán's inequality for ultraspherical polynomials \( P_n^{(\lambda)}(x) \) in the form

\[ \Delta_n^{(\lambda)}(x) \geq 0, \quad (\lambda > -\frac{1}{2}, \ n \geq 1, \ |x| \leq 1). \quad (6) \]

In his work on explicit evaluations of Turán expressions, A. E. Danese\(^3\) has obtained a number of relations which substantially contain the proof
Characteristic Relations for Ultraspherical Polynomials

of Turan’s inequality for the concerned orthogonal polynomials. K. Venkata-
chaliengar and the present author have shown that if

\[ \Delta_n^{(\lambda)}(x) = [P_n^{(\lambda)}(x)]^2 - P_{n+1}^{(\lambda)}(x) P_{n-1}^{(\lambda)}(x), \]

then \( \frac{d}{dx} P_n^{(\lambda)}(x) \) can be represented as a numerical multiple of the
Wronskian of \( P_{n+1}^{(\lambda)}(x) \) and \( P_{n-1}^{(\lambda)}(x) \) and have also obtained a series
representation of positive functions (apart from a multiplier) for \( \frac{d}{dx} \Delta_n^{(\lambda)}(x) \).

The aim of the present note is to point out that these several relations
employed or encountered in the course of proving Turán’s inequality for
\( P_n^{(\lambda)}(x) \), are characteristic for the ultraspherical polynomials.

**Theorem 1.**—If

\[ \Delta_n(x) = [P_n(x)]^2 - P_{n+1}(x) P_{n-1}(x), \]  

then

\[ \frac{d^2}{dx^2} \Delta_n(x) = -\frac{2}{n(n+1)} \left[ \frac{d}{dx} P_n(x) \right]^2. \]  

This relation is characteristic for Legendre polynomials, i.e., If \( \{f_n(x)\} \)
is a sequence of polynomials such that the degree of

\[ f_n(x) = n; \quad f_0(x) = 1, \quad f_1(x) = x \]

and

\[ \frac{d^2}{dx^2} \{[f_n(x)]^2 - f_{n+1}(x) f_{n-1}(x)\} = \frac{-2}{n(n+1)} \left[ \frac{d}{dx} f_n(x) \right]^2, \]  

then

\[ f_n(x) = P_n(x), \quad (n = 0, 1, 2, \ldots). \]

**Proof.**—We notice that

\[ f_0(x) = P_0(x), \quad f_1(x) = P_1(x). \]

Let us assume that

\[ f_r(x) = P_r(x), \quad 0 \leq r \leq n \]

and let

\[ f_{n+1}(x) = A_{n+1} x^{n+1} + \ldots. \]

Comparing the coefficients of \( x^{2n-2} \) on both sides of (9), we find that

\[ A_{n+1} = 2^{n+1} \left( \frac{n+3}{2} \right) [\Gamma(n+1)] (n+1)! \]
the highest coefficient in \( P_{n+1}(x) \), so that we may write

\[ f_{n+1}(x) = P_{n+1}(x) + g(x) \]

where \( g(x) \) is a polynomial of degree \( \leq n \). This substitution for \( f_{n+1}(x) \) in (9) leads to the equation

\[ \frac{d^2}{dx^2} [g(x) P_{n-1}(x)] = 0, \quad (n = 1, 2, 3, \ldots). \]

This second order differential equation has the solution

\[ g(x) = (c_1 x + c_2)/P_{n-1}(x), \quad (n \geq 1) \]

which is clearly not compatible with the polynomial nature of \( g(x) \). We conclude that \( c_1 = c_2 = 0 \) or \( g(x) = 0 \) and thus prove the theorem.

**Theorem 2**.—If

\[ \Delta^n (x) = [P_n^{(\lambda)}(x)]^2 - P_{n+1}^{(\lambda)}(x) P_{n-1}^{(\lambda)}(x) \]

then

\[ \frac{d}{dx} \Delta^n (x) = \frac{1 - \lambda}{n + \lambda} \left| \begin{array}{cc} P_{n+1}^{(\lambda)}(x) & P_{n-1}^{(\lambda)}(x) \\ dP_{n+1}^{(\lambda)}(x) & dP_{n-1}^{(\lambda)}(x) \end{array} \right| \]

and the relation is characteristic for the ultraspherical polynomials \( P_n^{(\lambda)}(x) \). If \( \{f_n^{(\lambda)}(x)\} \) is a sequence of polynomials such that the degree of

\[ f_n^{(\lambda)}(x) = n; \quad f_0^{(\lambda)}(x) = 1, \quad f_1^{(\lambda)}(x) = 2\lambda x \]

and

\[ \frac{d}{dx} \left\{ [f_n^{(\lambda)}(x)]^2 - f_{n+1}^{(\lambda)}(x) f_{n-1}^{(\lambda)}(x) \right\} \]

\[ \frac{1 - \lambda}{n + \lambda} \left| \begin{array}{cc} f_{n+1}^{(\lambda)}(x) & f_{n-1}^{(\lambda)}(x) \\ d f_{n+1}^{(\lambda)}(x) & d f_{n-1}^{(\lambda)}(x) \end{array} \right| \]

then

\[ f_n^{(\lambda)}(x) = P_n^{(\lambda)}(x), \quad (n = 0, 1, 2, \ldots). \]

**Proof**.—We take \( \lambda > -\frac{1}{2} \) but omit the case \( \lambda = 1 \) for then we have \( \Delta_n^{(1)}(x) = 1 \) and both sides of (11) are zero. As in the proof of Theorem 1, if we take

\[ f_r^{(\lambda)}(x) = P_r^{(\lambda)}(x), \quad 0 \leq r \leq n \]
Characteristic Relations for Ultraspherical Polynomials

and set

\[ f_{n+1}^{(\lambda)}(x) = A_{n+1}^{(\lambda)} x^{n+1} + \ldots, \]

by comparing the highest coefficients in (12), we notice that

\[ f_{n+1}^{(\lambda)}(x) = P_{n+1}^{(\lambda)}(x) + g^{(\lambda)}(x), \]

where \( g^{(\lambda)}(x) \) is a polynomial of degree \( \leq n \). Employing this relation in (12) and using (11), we arrive at the relation

\[ (n + 1) g^{(\lambda)}(x) \frac{d}{dx} P_{n-1}^{(\lambda)}(x) \]

\[ + (n + 2\lambda - 1) \frac{d g^{(\lambda)}(x)}{dx} P_{n-1}^{(\lambda)}(x) = 0. \quad (13) \]

We may evaluate \( g^{(\lambda)}(x) \) by direct integration. Otherwise, remembering that \( P_{n-1}^{(\lambda)}(x) \) and \( d/dx P_{n-1}^{(\lambda)}(x) \) have no common factor, we conclude from (13) that \( g^{(\lambda)}(x) = a P_{n-1}^{(\lambda)}(x) \). It follows then that

\[ a [(n + 1) + (n + 2\lambda - 1)] P_{n-1}^{(\lambda)}(x) \frac{d}{dx} P_{n-1}^{(\lambda)}(x) = 0. \]

Thus we are forced to accept that \( a = 0 \) and the theorem is proved.

**THEOREM 3.**—The difference equation

\[ \frac{d}{dx} A_{n}^{(\lambda)}(x) = \frac{(n + 2\lambda - 2)(n + 2\lambda - 3)}{n(n + 1)} \frac{d}{dx} A_{n-1}^{(\lambda)}(x) \]

\[ - \frac{8(n + \lambda - 1)(1 - \lambda)}{n(n + 1)} x [P_{n-1}^{(\lambda)}(x)]^2, \quad (14) \]

and the series representation

\[ \frac{d}{dx} A_{n}^{(\lambda)}(x) = \frac{8(\lambda - 1)[n + 2\lambda - 1]}{(n + 1)!} \]

\[ \times x \sum_{\kappa=1}^{[n+1/2]} \frac{(n - 2\kappa + 1)! (n - 2\kappa + \lambda + 1)}{|n - 2\kappa + 2\lambda + 1|} \]

\[ \times [P_{n-2\kappa+1}^{(\lambda)}(x)]^2 \]  

(15)

are characteristic relations for the ultraspherical polynomials \( P_{n}^{(\lambda)}(x) \). The proof runs exactly on the same lines as in Theorem 2. Here also, when \( \lambda = 1 \) both the members of the equation reduce to zero.
THEOREM 4.—The formula
\[
(1 - x^2) \left[ \frac{d}{dx} P_n^{(\lambda)}(x) \right]^2 - \frac{d}{dx} P_{n+1}^{(\lambda)}(x) \frac{d}{dx} P_{n-1}^{(\lambda)}(x) \right] = n(n + 2\lambda) [P_n^{(\lambda)}(x)]^2 - (n + 1)(n + 2\lambda - 1) P_{n+1}^{(\lambda)}(x)
\times (x) P_{n-1}^{(\lambda)}(x)
\]
(16)
is characteristic for the ultraspherical polynomials. Here again the proof is as in that of Theorem 2. In the same way we may show that the relation
\[
(n + 1)(n + 2\lambda - 1) [P_n^{(\lambda)}(x)]^2 - P_{n+1}^{(\lambda)}(x) P_{n-1}^{(\lambda)}(x)
\]
\[
\sum_{i=0}^{n} \sum_{j=1}^{n+1} \binom{j + \lambda}{j + 1} \binom{h_j^{(\lambda)}}{h_{j+1}^{(\lambda)}} [P_i^{(\lambda)}(x)]^2
\]
+ (2\lambda - 1) [P_n^{(\lambda)}(x)]^2,
\]
where
\[
h_{\kappa}^{(\lambda)} = \sqrt{\frac{\pi}{2\lambda + 1}} \left( \frac{2\lambda + 1}{2\lambda} \right) \binom{\kappa + \lambda}{\kappa + \lambda}
\]
is characteristic for the ultraspherical polynomials.

REFERENCES