THE CHARACTERISTIC ROOTS OF THE PRODUCT OF MATRICES

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1. Introduction

Let \( A = (a_{rs}) \) be an arbitrary square matrix of order \( n \) and \( c(A) \) a characteristic root of \( A \). Hirsch\(^5\) proved that

\[
|c(A)| \leq n \cdot \max_{1 \leq r, s \leq n} |a_{rs}|, \tag{1.1}
\]

and Browne\(^3\) obtained that

\[
|c(A)| \leq (R + T)/2, \tag{1.2}
\]

where

\[
R = \max_{(r)} \max_{(s)} \sum_{s=1}^{n} |a_{rs}|, \quad T = \max_{(s)} \sum_{r=1}^{n} |a_{rs}|.
\]

Further, Browne\(^3\) showed that

\[
s \leq |c(A)| \leq G, \tag{1.3}
\]

where \( s \) and \( G \) are the smallest and greatest characteristic roots of \( AA' \).

(1.1), (1.2) and other results established by several authors (for a list of references see [1]) give upper limits to \( |c(A)| \) in terms of the elements of \( A \), while (1.3) gives limits to \( |c(A)| \) in terms of the characteristic roots of \( AA' \). In the present note an upper limit for \( |c(A)| \) has been found in terms of the characteristic roots of the associated Hermitian matrices \( (A + A')/2 \) and \( (A - A')/2i \), and this result has been generalised for the characteristic roots of the product matrix \( AB \) of two \( n \)-square real or complex matrices \( A \) and \( B \). An upper limit for \( |c(AB)| \) has also been found in terms of the elements of \( A \) and \( B \). This result contains (1.2) as a particular case.

In establishing these results we use the well-known facts, see [4; p. 149], concerning the operator norm

\[
\|A\| = \max. \|Ay\| \text{ for } \|y\| = 1; \text{ when } \|y\| = \sqrt{y^*y}.
\]

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In what follows $R(A) = \max \sigma R_A = \max \sigma + \sum_{s=1} |a_{rs}|$ and $T(A) = \max \sigma T_A = \max \sigma + \sum_{s=1} |a_{rs}|$.

2. Bound Theorem for $c(A)$

Theorem 1.—Let $c(A)$ be an arbitrary characteristic root of $A$. Then

$$|c(A)| \leq \max |c\left(\frac{A + \bar{A}'}{2}\right) + \max |c\left(\frac{A - \bar{A}'}{2i}\right)|.$$ (2.1)

Proof.—Any square matrix $A = (A + \bar{A}')/2 + (A - \bar{A}')/2 = M + iN$, say, where $M$ and $N$ are Hermitian. Corresponding to an arbitrary characteristic root $\lambda$ of $A$, there exists a non-zero column vector $x$, such that $\lambda x = Ax$. Then

$$|\lambda| \cdot \|x\| = \|Ax\| = \|(M + iN)x\|$$

$$\leq \|M + iN\| \cdot \|x\|,$$

where for any square matrix $B$, $\|B\|^2 = \max \|Bx\| = \max (Bx, Bx)$

$$= \max (\bar{B}'B) = c_{\text{max}} (B'B),$$

$c_{\text{max}}$ denoting the greatest characteristic root, and in particular if $B$ is Hermitian $\|B\| = \max |c(B)|$.

Since

$$\|x\| = \sqrt{\lambda x > 0},$$

we have

$$|\lambda| \leq \|M + iN\|$$

$$\leq \|M\| + \|N\|$$

$$= \max |c\left(\frac{A + \bar{A}'}{2}\right)| + \max |c\left(\frac{A - \bar{A}'}{2i}\right)|.$$

This establishes (2.1).

It may be observed that (1.3) and (2.1) give the same upper limit for $|c(A)|$ when $A$ is Hermitian or skew-Hermitian. There are some other matrices also, for example $\begin{pmatrix} 1 & i \\ -3i & -1 \end{pmatrix}$, which are not of some special type but for which (1.3) and (2.1) give the same upper limit for $|c(A)|$. The limit for $|c(A)|$ given by (2.1) is not better than that given by (1.3), but,
in general, it is easier to find the characteristic roots of \((A + \bar{A}')/2\) and 
\((A - \bar{A}')/2i\) than those of \(A\bar{A}'\).

3. Bound Theorems for \(c(AB)\)

We now consider two theorems which give upper limits to the absolute value of an arbitrary characteristic root of the product matrix \(AB\).

**Theorem 2.** Let \(c(AB)\) denote an arbitrary characteristic root of \(AB\). Then

\[
|c(AB)| \leq \left[ \max \left| c\left(\frac{A + A'}{2}\right) \right| + \max \left| c\left(\frac{A - A'}{2i}\right) \right| \right] 
\times \left[ \max \left| c\left(\frac{B + B'}{2}\right) \right| + \max \left| c\left(\frac{B - B'}{2i}\right) \right| \right].
\]  

(3.1)

**Proof.** Any square matrix \(A = (A + \bar{A}')/2 + (A - \bar{A}')/2 = M + iN\), say, and \(B = (B + \bar{B}')/2 + (B - \bar{B}')/2 = R + iS\), say, where \(M, N, R\) and \(S\) are Hermitian matrices. Thus \(AB = (M + iN)(R + iS)\).

If \(\mu\) is an arbitrary characteristic root of \(AB\), there exists a non-zero column vector \(x\), such that 
\[
\mu x = ABx
\]
so that
\[
|\mu| |x| = |\mu x|
= |ABx|
\leq \|(M + iN)|| \cdot \|(R + iS)|| \cdot \|x\|.
\]

Since
\[
\|x\| > 0,
\]
we have
\[
|\mu| \leq ||M + iN|| \cdot ||R + iS||
\leq (||M|| + ||N||)(||R|| + ||S||)
= (\max |c(M)| + \max |c(N)|)(\max |c(R)| + \max |c(S)|)
\]
since \(M, N, R\) and \(S\) are Hermitian matrices.

This establishes (3.1). It is easy to see that (3.1) can be generalised to the case of the product of any finite number of matrices \(A_1, A_2, \ldots, A_k\) of the same order. The result in the generalised form is

\[
|c \left( \prod_{r=1}^{k} A_r \right)| \leq \prod_{r=1}^{k} \left\{ \max \left| c\left(\frac{A_r + A_{r}'}{2}\right) \right| + \max \left| c\left(\frac{A_r - A_{r}'}{2i}\right) \right| \right\}.  
\]  

(3.2)
The Characteristic Roots of the Product of Matrices  

THEOREM 3.—Let \( c(AB) \) denote an arbitrary characteristic root of \( AB \). Then

\[
| c(AB) | \leq \frac{R(A) + T(A)}{2} \cdot \frac{R(B) + T(B)}{2}.
\]

(3.3)

Proof.—If \( \mu \) is an arbitrary characteristic root of \( AB \), there exists a non-zero column vector \( x \), such that \( \mu x = ABx \).

This gives, as we have seen in the proof of the previous theorem,

\[
| \mu | \leq \| AB \| \leq \| A \| \cdot \| B \| \leq \frac{R(A) + T(A)}{2} \cdot \frac{R(B) + T(B)}{2},
\]

for, by a known result [3; p. 708]

\[
\| A \| = | c_{\text{max}}(AA') | \leq \frac{R(A) + T(A)}{2}.
\]

This completes the proof of Theorem 3. It may be observed that (3.3) can be generalized to the case of the product of a finite number of matrices \( A_1, A_2, \ldots, A_k \) of the same order. The result in the generalized form is

\[
| c\left(\prod_{r=1}^{k} A_r\right) | \leq \prod_{r=1}^{k} \frac{R(A_r) + T(A_r)}{2}.
\]

(3.4)

4. PARTICULAR CASES OF THEOREM 3

(i) If we put \( B = I \), (3.3) reduces to

\[
| c(A) | \leq \frac{R(A) + T(A)}{2},
\]

a result due to Browne,3 quoted earlier.

(ii) If \( A \) and \( B \) are Hermitian or skew-Hermitian matrices, then \( R_r(A) = T_r(A) \) and \( R_r(B) = T_r(B) \) for \( r = 1, 2, \ldots, n \), which means \( R(A) = T(A) \) and \( R(B) = T(B) \). Hence for Hermitian or skew-Hermitian matrices \( A \) and \( B \)

\[
| c(AB) | \leq R(A) R(B).
\]

(3.5)

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REFERENCES

1. Brauer, A. 

2. Browne, E. T. 

3. ———— 

4. Halmos, P. R. 

5. Hirsch, A. 