THE GREEN'S FUNCTIONS FOR EQUATIONS OF PARTICLES OF ARBITRARY SPIN

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THE wave equation for a particle of arbitrary spin can be written in the form (Bhabha, 1949)

$$(p_k\, a^k + \chi)\, \psi = 0, \tag{1}$$

where

$$p_k = -\, i\, \frac{\partial}{\partial x^k}\ ;\quad x^0\,(= t),\ x^1,\ x,^2\ x^3$$

are the co-ordinates of a point in space-time; a^k are $\nu \times \nu$ matrices, and ψ is a ν-component wave function. χ is a constant related to the masses of the particle described by (1). a^0 satisfies, in the general case, the minimal equation

$$(a^0)^m\, \{(a)^2 - a_1{}^2\}\, \{(a^0)^2 - a_2{}^2\} \ldots \{(a^0)^2 - a_n{}^2\} = 0. \tag{2}$$

Then also

$$\mathrm{P}^m\, (\mathrm{P}^2 - a_1{}^2 p^2)\ (\mathrm{P}^2 - a_2{}^2 p^2) \ldots (\mathrm{P}^2 - a_n{}^2\ p^2) = 0, \tag{3}$$

where

$$\mathrm{P} = p_k\ a^k$$

and

$$p^2 = p_k\ p^k.$$

Equation (1) will describe a particle which has states of masses χ/a_1, $\chi/a_2, \ldots, \chi/a_n$.

The commutation relations for the particle field have been found by Udgaonkar (1953). They are

$$[\psi\,(x),\ \psi^{\dagger}\,(x')\,\mathrm{D}] = -\, i\mathrm{S}\,(x - x'), \tag{4}$$

where

$$\mathrm{S}\,(x) = \sum_{r=1}^{n} \frac{(-\,1)^m}{\chi^{2n+m-2}\, a_r{}^2 \left(1 - \frac{a_1{}^2}{a_r{}^2}\right)\left(1 - \frac{a_2{}^2}{a_r{}^2}\right) \cdots \left(1 - \frac{a_{r-1}{}^2}{a_r{}^2}\right)\left(1 - \frac{a_{r+1}{}^2}{a_r{}^2}\right) \cdots \left(1 - \frac{a_n{}^2}{a_r{}^2}\right)}$$

$$\times\, \mathrm{P}^m \left(\mathrm{P}^2 - \frac{\chi^2 a_1{}^2}{a_r{}^2}\right) \cdots \left(\mathrm{P}^2 - \frac{\chi^2 a_{r-1}{}^2}{a_r{}^2}\right) (\mathrm{P} - \chi) \left(\mathrm{P}^2 - \frac{\chi^2 a_{r+1}{}^2}{a_r{}^2}\right) \cdots \left(\mathrm{P}^2 - \frac{\chi^2 a_n{}^2}{a_r{}^2}\right)$$

$$\times\, \Delta_r\,(x)\,; \tag{5}$$

here $\Delta_r(x)$ is the well-known invariant function of Jordan and Pauli for mass χ/a_r. [A, B] stands for AB + BA or AB − BA according as the charge density or the energy density is positive definite.

We first prove that S(x) defined by (5) satisfies

$$(P + \chi)\, S(x) = 0.$$

Now

$$(P + \chi)\, S(x) = \sum_{r=1}^{n} A_r P^m \left(P^2 - \frac{\chi^2 a_1^2}{a_r^2}\right) \cdots \left(P^2 - \frac{\chi^2 a_{r-1}^2}{a_r^2}\right)(P^2 - \chi^2) \times \left(P^2 - \frac{\chi^2 a_{r+1}^2}{a_r^2}\right) \cdots \left(P^2 - \frac{\chi^2 a_n^2}{a_r^2}\right) \Delta_r(x) \tag{6}$$

where we have put

$$A_r = \frac{(-1)^m}{\chi^{2n+m-2} a_r^2 \left(1 - \frac{a_1^2}{a_r^2}\right)\left(1 - \frac{a_2^2}{a_r^2}\right) \cdots \left(1 - \frac{a_{r-1}^2}{a_r^2}\right)\left(1 - \frac{a_{r+1}^2}{a_r^2}\right) \cdots \left(1 - \frac{a_n^2}{a_r^2}\right)}. \tag{7}$$

Since $\Delta_r(x)$ satisfies the equation

$$\left(p^2 - \frac{\chi^2}{a_r^2}\right)\Delta_r(x) = 0, \tag{8}$$

we can replace χ^2/a_r^2 by p^2 in equation (5); we get thereby

$$(P + \chi)\, S(x) = \Sigma A_r P^m (P^2 - a_1^2 p^2)(P^2 - a_2^2 p^2) \cdots (P^2 - a_n^2 p^2)\, \Delta_r(x)$$
$$= 0;$$

the last step follows from equation (3).

Let

$$\bar{\Delta}_r(x) \equiv -\tfrac{1}{2}\, \xi(x)\, \Delta_r(x), \tag{9}$$

where

$$\xi(x) = +1 \text{ for } x_0 > 1$$
$$= -1 \text{ for } x_0 < 1.$$

Then [*cf.* Pauli, 1950]

$$\left(p^2 - \frac{\chi^2}{a_r^2}\right) \bar{\Delta}_r(x) = -\,\delta(x), \tag{10}$$

where

$$\delta(x) = \delta(x^0)\, \delta(x^1)\, \delta(x^2)\, \delta(x^3).$$

Let $\bar{S}(x)$ denote the function obtained from $S(x)$ by replacing the $\Delta_r(x)$ by $\bar{\Delta}_r(x)$, *viz.*,

$$\bar{S}(x) = \Sigma A_r P^m \left(P^2 - \frac{\chi^2 a_1^2}{a_r^2}\right) \cdots \left(P^2 - \frac{\chi^2 a_{r-1}^2}{a_r^2}\right)(P - \chi) \times \left(P^2 - \frac{\chi a_{r+1}^2}{a_r^2}\right) \cdots \left(P^2 - \frac{\chi^2 a_r^2}{a_r^2}\right) \bar{\Delta}_r(x). \quad (11)$$

We shall now prove that

$$(P + \chi)\bar{S}(x) = -(-1)^m \frac{P^m}{\chi^m} \delta(x).$$

Operating on equation (11) from the left by $(P + \chi)$ we get

$$(P + \chi)\bar{S}(x) = \Sigma A_r P^m \left(P^2 - \frac{\chi^2 a_1^2}{a_r^2}\right) \cdots \left(P^2 - \frac{\chi^2 a_n^2}{a_r^2}\right) \bar{\Delta}_r(x)$$

$$= \Sigma A_r P^m \left\{P^{2n} + P^{2(n-1)}\left(-x^2 \frac{a_1^2}{a_r^2} - \chi^2 \frac{a_2^2}{a_r^2} - \cdots - \frac{\chi^2 a_n^2}{a_r^2}\right) + P^{2(n-2)}\left(\frac{\chi^2 a_1^2}{a_r^2} \frac{\chi^2 a_2^2}{a_r^2} + \frac{\chi^2 a_1^2}{a_r^2} \frac{\chi^2 a_3^2}{a_r^2} + \cdots\right) + \cdots + (-1)^n \left(\frac{\chi^2}{a_r^2}\right)^n a_1^2 a_2^2 \cdots a_n^2 \right\} \bar{\Delta}_r(x). \quad (12)$$

Now from equation (3)

$$P^m \{P^{2n} + P^{2(n-1)}(-a_1^2 p^2 - a_2^2 p^2 - \cdots - a_n^2 p^2) + P^{2(n-2)}(a_1^2 p^2 a_2 p^2 + a_1^2 p^2 a_3^2 p^2 + \cdots) + \cdots + (-1)^n p^{2n} a_1^2 a_2^2 \cdots a_n^2\} = 0.$$

Substituting for P^{2n+m} from this equation in equation (12) we obtain

$$(P + x)\,\bar{S}(x) = \Sigma A_r P^m \left[P^{2(n-1)} \left\{a_1^2 \left(p^2 - \frac{\chi^2}{a_r^2}\right) + a_2^2 \left(p^2 - \frac{\chi^2}{a_r^2}\right) + \cdots + a_n^2 \left(p^2 - \frac{\chi^2}{a_r^2}\right)\right\} + P^{2(n-2)} \left\{a_1^2 a_2^2 \left(\frac{\chi^4}{a_r^4} - p^4\right) + \left(\frac{\chi^4}{a_r^4} - p^4\right) \times a_1^2 a_3^2 + \cdots\right\} + \cdots + (-1)^n a_1^2 a_2^2 \cdots a_n^2 \left\{\left(\frac{\chi^2}{a_r^2}\right)^n - p^{2n}\right\}\right] \bar{\Delta}_r(x).$$

On the right-hand side of this equation $(p^2 - \chi^2/a_r^2)$ is a common factor. In view of the inhomogeneous equation (10) satisfied by $\bar{\Delta}_r(x)$, we obtain

$$(P + x)\,\bar{S}(x) = -\sum_r A_r P^m \Big[P^{2(n-1)} (a_1^2 + a_2^2 + \cdots + a_n^2)$$

$$+ P^{2(n-2)} (- a_1^2 a_2^2 - a_1^2 a_3^2 - \cdots) \left(p^2 + \frac{\chi^2}{a_r^2}\right) + \cdots$$

$$+ (-1)^n a_1^2 a_2^2 \cdots a_n^2 \Big\{ - p^{2(n-1)} - p^{2(n-2)} \frac{\chi^2}{a_r^2} - p^{2(n-3)} \left(\frac{\chi^2}{a_r^2}\right)^2$$

$$- \cdots - p^2 \left(\frac{\chi^2}{a_r^2}\right)^{n-2} - \left(\frac{\chi^2}{a_r^2}\right)^{n-1} \Big\} \Big] \delta(x). \quad (13)$$

Now the numbers A_r defined by (7) satisfy the following identities [Bhabha, 1950; *cf.* also the Appendix]:

$$\sum_{r=0}^{n} A_r = 0; \quad \sum \frac{1}{a_r^2} A_r = 0; \quad \cdots \sum \frac{1}{a_r^{2(n-2)}} A_r = 0; \quad (14\,a)$$

$$\sum \frac{1}{a_r^{2(n-1)}} A_r = (-1)^{m+n-1} \frac{1}{\chi^{2n+m-2}} \frac{1}{a_1^2 a_2^2 \cdots a_r^2}. \quad (14\,b)$$

Hence on the right-hand side of equation (13) all terms, excepting the last one which contains $(\chi^2/a_r^2)^{n-1}$, vanish. The contribution due to this term is

$$-(-1)^m \frac{P^m}{\chi^m} \delta(x)$$

so that

$$(P + \chi)\,\bar{S}(x) = -(-1)^m \frac{P^m}{\chi^m} \delta(x) \quad (15)$$

We now define

$$G(x) = \Big[\bar{S}(x) + \frac{(-1)^m}{\chi^m} \{ P^{m-1} - \chi P^{m-2} + \chi^2 P^{m-3} - \cdots$$

$$+ (-1)^{m-1} \chi^{m-1} \} \delta(x) \Big]. \quad (16)$$

Then

$$(P + \chi)\,G(x) = \Big[-(-1)^m \frac{P^m}{\chi^m} \delta(x) + \frac{(-1)^m}{\chi^m} \{ P^m$$

$$+ (-1)^{m-1} \chi^m \} \; \delta(x). \Big] = -\delta(x).$$

Hence $G(x)$ is the Green's function for equation (1).

The positive and negative frequency parts $S^+(x)$ and $S^-(x)$ respectively, of $S(x)$ are obtained by replacing the Δ_r in $S(x)$ by Δ_r^+ and Δ_r^-, which are defined by

$$\Delta_r^+ = -\frac{2}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{2\omega}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)},$$

$$\Delta_r^- = \frac{i}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{2\omega}\, e^{i(\mathbf{k}\cdot\mathbf{x}+\omega t)}, \quad \omega = +\sqrt{\mathbf{k}^2 + \frac{\chi^2}{a_r^2}}\,.$$

The function $S^{(1)} \equiv i(S^+ - S^-)$ is also obtained by replacing the Δ_r in S by $\Delta_r^{(1)}$. All these functions satisfy the homogeneous wave equation. The Feynman function S_F is defined by (*cf.* Pauli, 1950)

$$S_F(x) = S^{(1)}(x) - 2i\,G(x) \tag{17}$$

and satisfies the equation

$$(P + \chi)\,S_F(x) = 2i\,\delta(x). \tag{18}$$

Putting

$$\Delta_{Fr} = \Delta_r^{(1)} - 2i\,\bar{\Delta}_r \tag{19}$$

we obtain

$$S_F(x) = \sum_{r=1}^{n} A_r P^m \left(P^2 - \frac{\chi^2 a_1^2}{a_r^2}\right)\cdots\left(P^2 - \frac{\chi^2 a^2_{r-1}}{a_r^2}\right)(P - \chi)\left(P^2 - \frac{\chi^2 a^2_{r+1}}{a_r}\right)$$
$$\cdots\left(P^2 - \frac{\chi^2 a_r^2}{a_r^2}\right)\Delta_{Fr}(x)$$
$$-\,2i\frac{(-1)^m}{\chi^m}\{P^{m-1} - \chi P^{m-2} + \chi^2 P^{m-3} - \cdots + (-1)^{m-1}\chi^{m-1}\}\,\delta(x). \tag{20}$$

To get the momentum space representation of $S_F(x)$ we make use of the following Fourier representations

$$\Delta_{Fr}(x) = \frac{-2i}{(2\pi)^4}\int^{C} d^4p\, \frac{e^{ipx}}{-p^2 + \frac{\chi^2}{a_r^2}}, \tag{21}$$

$$\delta(x) = \frac{1}{(2\pi)^4}\int d^4p\, e^{ipx}, \tag{22}$$

$$px = p^0x^0 - p^1x^1 - p^2x^2 - p^3x^3,$$

$$p^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2.$$

In the first equation above the integration over p^1, p^2, p^3 is along the real axes, and over p^0 is along the real axis with two small detours one below the point $-\{(p')^2 + (p^2)^2 + (p^3)^2 + \chi^2/a_r^2\}^{\frac{1}{2}}$ and the other above the point $+\{(p')^2 + (p^2)^2 + (p^3)^2 + \chi^2/a_r^2\}^{\frac{1}{2}}$ in the p^0 plane. We obtain

$$S_F(x) = \frac{2i}{(2\pi)^4} \int d^4p\, S_F(p)\, e^{ipx}, \tag{23}$$

where

$$S_F(p) = -\Big[\sum_r A_r P^m \Big(P - \chi^2 \frac{a_1^2}{a_r^2}\Big) \cdots \Big(P^2 - \chi^2 \frac{a_{r-1}^2}{a_r^2}\Big)(P - \chi)$$
$$\times \Big(P^2 - \chi^2 \frac{a^2_{r+1}}{a_r^2}\Big) \cdots \Big(P^2 - \chi^2 \frac{a_n^2}{a_r^2}\Big) \frac{1}{-p^2 + \dfrac{\chi^2}{a_r^2}}$$
$$+ \frac{(-1)^m}{\chi^m}\{P^{m-1} - \chi P^{m-2} + \chi^2 P^{m-3} - \cdots + (-1)^{m-1}\chi^{m-1}\}\Big]. \tag{24}$$

Making use of the identities (14) it can easily be proved that

$$(P + \chi)\; S_F(P) = 1,$$

so that

$$S_F(x) = \frac{2i}{(2\pi)^4} \int d^4 p\, (P + \chi)^{-1}\, e^{ipx}. \tag{25}$$

It should be noted that this is only a formal way of writing $S_F\,(x)$, since $(P + \chi)$ is a singular operator for certain values of p_k, because $(P + \chi)\,\psi = 0$ for non-vanishing ψ, and $(P + \chi)^{-1}$ then does not exist. At the poles of $(P + \chi)^{-1}$ the path of integration in (25) has to be chosen as stated above.

The function $S_F(x)$ given above is twice the propagation function K_+ as defined in the theory of Feynman (1949). Hence the graphical techniques of Feynman and Dyson can be applied in the general case of particles of arbitrary spins if one uses the function $\frac{1}{2}\, S_F\,(x)$ for the internal particle lines.

In conclusion, I should like to express my sincere thanks to Professor H. J. Bhabha for his interest in the work.

APPENDIX

The following proof of identities (14 *a*, *b*) is due to Dr. K. G. Ramanathan:

Let

$$f(x) = (x - a_1^2)\,(x - a_2^2) \cdots (x - a_n^2) \tag{A 1}$$

and let $\phi(x)$ be a polynomial of degree $< n$. Then

$$\frac{\phi(x)}{f(x)} = \frac{A_1}{x - a_1^2} + \frac{A_2}{x - a_2^2} + \cdots + \frac{A}{x - a_n^2}, \tag{A 2}$$

where

$$A_r = \frac{\phi(a_r^2)}{(a_r^2 - a_1^2)(a_r^2 - a_2^2)\cdots(a_r^2 - a_{r-1}^2)(a_r^2 - a_{r+1}^2)\cdots(a_r^2 - a_n^2)}. \tag{A 3}$$

Let

$$\phi(x) = x^{n-2}$$

then

$$x^{n-2} = \sum_{r=1}^{n} \frac{a_r^{2(n-2)}}{(a_r^2 - a_1^2)\cdots(a_r^2 - a_{r-1}^2)\,(a_r^2 - a_{r+1}^2)\cdots(a_r^2 - a_n^2)} \times (x - a_1^2)\,(x - a_2^2)\cdots(x - a_{r-1}^2)\,(x - a_{r+1}^2)\cdots(x - a_n^2). \tag{A 4}$$

Equating the coefficient of x^{n-1} on the right-hand side of this equation to zero we get

$$\Sigma \frac{a_r^{2(n-2)}}{(a_r^2 - a_1^2)\,(a_r^2 - a_2^2)\cdots(a_r^2 - a_{r-1}^2)\,(a_r^2 - a_{r+1}^2)\cdots(a_r^2 - a_n^2)} = 0. \tag{A 5}$$

This is the first of the identities (14 *a*) of the text. Similarly, taking $\phi(x) = x^{n-3}, x^{n-4}, \ldots, x$, we obtain other identities of set (14 *a*).

To prove (14 *b*) take $\phi(x) = 1$. Then

$$1 = \Sigma \frac{1}{(a_r^2 - a_1^2)\cdots(a_r^2 - a_{r-1}^2)\,(a^2 - a_{r+1}^2)\cdots(a_r^2 - a_n^2)} \times (x - a_1^2)\,(x - a_2^2)\cdots(x - a_{r-1}^2)\,(x - a_{r+1}^2) \cdots (x - a_n^2). \tag{A 6}$$

The constant term on the right-hand side must equal 1. Hence

$$1 = (-1)^{n-1} \sum_r \frac{a_1^2 a_2^2 \cdots a_n^2}{a_r^2 (a_r^2 - a_1^2) \cdots (a_r^2 - a_{r-1}^2)(a_r^2 - a_{r+1}^2) \cdots (a_r^2 - a_n^2)}, \tag{A 7}$$

which is the identity (14 *b*).

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