REMARKS ON A PROBLEM IN SYMMETRIC FUNCTIONS

By G. S. Mahajani, F.A.Sc.

(*Vice-Chancellor, Delhi University, Delhi*)

AND

V. R. Thiruvenkatachar

(*Institute of Armament Studies, Kirkee, Poona*)

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1. Introduction

In a former paper (*Proc. Ind. Acad. Sci.*, 1952, **35 A**, referred to hereafter as **p**) we considered the following problem: Given that the set of variables $(x_1, x_2, \ldots, x_n)$ satisfies the conditions:

$$(a)\ 0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$$

$$(b)\ \Sigma x_1 = k_1$$

$$\Sigma x_1 x_2 = k_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Sigma x_1 \ldots x_{n-1} = k_{n-1}$$

where $k_1, k_2, \ldots k_{n-1}$ are fixed but unspecified constants.

(*c*) Maximum and minimum values of the successive x's occur alternately, *i.e.*, if x_1 has its maximum value, x_2 has its minimum value, x_3 its maximum value and so on.

To find, under these conditions, n numbers $(\alpha_1, \alpha_2 \ldots, \alpha_n)$ such that

$$0 \leqslant x_1 \leqslant \alpha_1 \leqslant x_2 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_{n-1} \leqslant x_n \leqslant \alpha_n.$$

It was shown in **p** that the α's are the roots of certain Tschebyscheff polynomials and that they possess certain symmetry properties.

The object of the present note is to offer some supplementary remarks indicating an alternative formulation of the problem as a problem of conditional maxima and minima. This formulation shows that the original problem is but one of a class of similar problems, all of which are governed by the same extremising condition and that the "two extreme solutions" considered in the previous paper (**p**, p. 212) are indeed such. We give solutions for some of these other problems and also present some additional considerations regarding the original problem.

2. NEW FORMULATION OF THE PROBLEM

Given that

$$\alpha_0 \leqslant x_1 \leqslant \alpha_1 \leqslant x_2 \leqslant \alpha_2 \ldots \leqslant \alpha_{n-1} \leqslant x_n \leqslant \alpha_n \tag{1}$$

$$\left.\begin{array}{l} \Sigma x_1 = k_1 \\ \Sigma x_1 x_2 = k_2 \\ \cdots\cdots\cdots\cdots \\ \Sigma x_1 x_2 \ldots x_{n-1} = k_{n-1} \end{array}\right\} \tag{2}$$

To make $\phi = x_1 x_2 \ldots x_n = k_n$ an extremum.

We assume that no two of the α's in (1) can be equal, for if two of the α's are equal, say $\alpha_r = \alpha_{r-1}$, then x_r would be reduced to a constant and the problem would be reduced to one in $n - 1$ variables.

By the Lagrange method of undetermined multipliers it is readily seen that the stationary values of ϕ occur when

$$\Pi (x_r - x_s) = 0 \tag{3}$$

i.e., extreme values of ϕ occur when any two of the x's are equal and, on account of (1), this can happen only when

$$x_{r-1} = x_r, \qquad r = 2, \ldots n \tag{4}$$

Thus ϕ has $n - 1$ stationary values. It is now evident that our problem in **p** is that particular case corresponding to the maximum number of equal pairs of roots, *viz.*,

(i) $x_1 = x_2 = \alpha_1$, $x_3 = x_4 = \alpha_3$, etc.

or (ii) $x_1 = \alpha_0$, $x_2 = x_3 = \alpha_2$, $x_4 = x_5 = \alpha_4$, etc.

so that ϕ has only *two* stationary values. From (3) we see that ϕ is stationary when the polynomial

$$f(x) = x^n - k_1 x^{n-1} + k_2 x^{n-2} - k_3 x^{n-3} + \ldots \pm k_{n-1} x \mp k_n$$

has a pair of equal roots, the condition for which is the vanishing of the eliminant of

$$f\ (x) \equiv x^n - k_1 x^{n-1} + \ldots . \mp k_{n-1} x \pm k_n = 0$$

$$f'(x) \equiv n x^{n-1} - (n-1)\, k_1 x^{n-2} + \ldots . \mp k_{n-1} = 0$$

By Sylvester's theorem, the eliminant will be a relation of the form

$$\mathrm{F}\,(k_1,\ k_2,\ \ldots . k_{n-1},\ k_n) = 0 \tag{5}$$

in which k_1, k_2, $\ldots . k_{n-1}$ each occur to the degree n but k_n (or ϕ) occurs to the degree $n - 1$, corresponding to the $n - 1$ stationary values. These $(n - 1)$ stationary values will in general be different, but by a proper choice of k_1, k_2, $\ldots . k_{n-1}$, the eliminant (5) can be reduced to the form

$$(\phi - \lambda)^r\ (\phi - \mu)^s = 0, \qquad r + s = n - 1 \tag{6}$$

Conversely, if (5) reduces to the form (6), the k's are determined. The two stationary values given by (6) correspond to the 'two extreme solutions' referred to above. This offers a natural approach to our original problem but it seems difficult to carry out this programme in general.

3. Other Related Problems

The conditions for an extremum in the problem considered above can also be written as follows:—

$$\left.\begin{array}{ll} d\,(\Sigma x_1) & = 0 \\ d\,(\Sigma x_1 x_2) & = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ d\,(\Sigma x_1 x_2 \ldots . x_{n-1}) & = 0 \\ d\,(x_1 \ldots . x_n) & = 0 \end{array}\right\} \tag{7}$$

the last of which holds because of the required stationary character of the function, while the remaining equations are consequences of the constancy of the concerned functions. It will be observed that *exactly the same equations are obtained if, instead of making* $\phi = k_n$ *stationary, any one of the other k's is made an extremum, all the remaining* k*'s being fixed.* Thus the condition (3) for an extremum holds for all the n different problems:—

(i) any one k_r = extremum, $r = 1, 2, \ldots$ or n

(ii) all other k_s, $(n - 1)$ in number, $(s \neq r)$ fixed.

As in the original problem, we consider for each of the new problems only the case corresponding to the maximum number of double roots. By way of illustration we give the solutions for $n = 3$ and $n = 4$.

The Case $n = 3$

Problem	a_0	a_1	a_2	a_3
k_1 = Stationary k_2, k_3 fixed	$\frac{m+1}{2m}$	1	m	$\frac{m(m+1)}{2}$
k_2, = Stationary k_3, k_1 fixed	$\frac{2}{m+1}$	1	m	$\frac{2m^2}{m+1}$
k_3 = Stationary k_1, k_2 fixed	$\frac{3-m}{2}$	1	m	$\frac{3m-1}{2}$

The Case $n = 4$

Problem	a_0	a_1	a_2	a_3	a_4
k_1 = Stationary k_2, k_3 and k_4 fixed	$\frac{m^2(m-1)}{(m+1)(m-2)^2}$	1	m^2-1	$\frac{m^2(m^2-1)}{4-m^2}$	$\frac{m^2(m+1)}{(m-1)(m+2)^2}$
k_2 = Stationary k_3, k_4 and k_1 fixed	$(m^2-m+1)-(m+1)\sqrt{m^2+1}$	1	m	m^2	$(m^2-m+1)+(m+1)\sqrt{m^2+1}$
k_3 = Stationary k_4, k_1 and k_2 fixed	a	1	m	$\frac{m(3m-1)}{m+1}$	c
	where a, c are the roots of $(m+1)^2t^2 - 2(2m+1)(m^2-1)t + (3m-1)^2 = 0$				
k_4 = Stationary k_1, k_2 and k_3 fixed	$m-(m-1)\sqrt{2}$	1	m	$2m-1$	$m+(m-1)\sqrt{2}$

In our original problem (k_n-stationary) we took $a_0 = 0$. It is readily verified that the Polynomial Identity formulated for the case $a_0 \neq 0$ transforms itself into the very identity for the case $a_0 = 0$ (p, p. 217, 220) by the mere substitution:

$$\gamma v = t - a_0$$

$$a_r - a_0 = a_r'$$

$$\xi - a_0 = \gamma \xi'$$
$$\gamma \neq 0$$

Taking $\gamma = 1$, we see that, when $a_0 \neq 0$, the differences $(a_0 - a_1), (a_2 - a_0)\ldots,$ $(a_n - a_0)$ are the roots of the Tschebyscheff polynomial found in **p**. We might here draw attention to one important distinction between the problem of k_n stationary and the others corresponding to $k_r\,(r \neq n)$ stationary. In the former case, the ratios $(a_1 - a_0):(a_2 - a_0):\ldots\ldots:(a_n - a_0)$ are independent of any arbitrary parameter; this is *not* the case for the other problems.

4. Remarks on the Symmetry Property

For the solution of the original problem we established in **p** the following symmetry relations for any n:

$$a_0 + a_n = a_1 + a_{n-1} = a_2 + a_{n-2} = \cdots \tag{8}$$

It should be possible to prove these relations from the basic governing relations, without obtaining the explicit form of the solution.

Consider the case $n = 2m + 1$. We have then the polynomial identity (**p**, p. 217), *generalised* to the present case $a_0 \neq 0$,

$$\begin{aligned}\phi(t) &\equiv (t - a_0)\,(t - a_2)^2(t - a_4)^2\ldots\ldots(t - a_{2m})^2 \\ &\equiv (t - a_1)^2(t - a_3)^2\ldots\ldots(t - a_{2m-1})^2(t - \xi) + \\ &\qquad (\xi - a_0)\,(a_1 - a_0)^2\,(a_3 - a_0)^2\ldots\ldots(a_{2m-1} - a_0)^2.\end{aligned}$$

Since this is an identity, $a_0, a_2, a_4, \ldots\ldots a_{2m}$ as zeros of $\phi(t)$ are *uniquely* determined as functions of $a_1, a_3, \ldots\ldots a_{2m-1}, \xi$. By writing $\xi - t + a_0$ for t, the identity becomes, after a slight rearrangement

$$\begin{aligned}&(t - a_0)\,[t - (\xi + a_0 - a_1)]^2\,[t - (\xi + a_0 - a_3)]^2 \\ &\qquad [t - (\xi + a_0 - a_{2m-1})]^2 \\ &= [t - (\xi + a_0 - a_2)]^2\,[t - (\xi + a_0 - a_4)]^2 \\ &\qquad [t - (\xi + a_1 - a_{2m})]^2\,(t - \xi) \\ &\qquad + (\xi - a_0)\,(a_1 - a_0)^2\,(a_3 - a_0)^2\ldots(a_{2m-1} - a_0)^2\end{aligned}$$

Comparing this with the previous form we can see that the identity is satisfied if we write

$$\xi + \alpha_0 - \alpha_1 \quad = \alpha_{2m}$$

$$\xi + \alpha_0 - \alpha_3 \quad = \alpha_{2m-2}$$

$$\dots\dots\dots\dots\dots\dots$$

$$\xi + \alpha_0 - \alpha_{2m-1} = \alpha_2$$

Of course, $\xi = \alpha_{2m+1} = \alpha_n$.

On account of the above-mentioned uniqueness, it follows that the sought relations between $(\alpha_0, \alpha_2, \alpha_4 \dots \alpha_{2m})$ and $(\alpha_1, \alpha_3 \dots \alpha_{2m-1}, \xi)$ are just these, which are of course the desired symmetry relations. However a similar proof does not work for the case $n = 2m$.