THE MOTION OF FOUR RECTILINEAR VORTEX FILAMENTS

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1. INTRODUCTION

The motion of three parallel rectilinear vortex filaments of strengths $K_1$, $K_2$, $K_3$ in a perfect incompressible fluid extending to infinity originally examined by Gröbli (1877) has been again considered by Synge (1949) who has made an exhaustive study of the motion by representing the various configurations of vortices by points in trilinear co-ordinates on a representative plane. A similar study of all the motions of four vortex filaments presents great difficulty owing to the very large number of cases to be dealt with and the lack of corresponding geometrical representation adequate for classifying all the motions. The problem of four vortex filaments which form fixed configurations, i.e., configurations with fixed sides and diagonals was previously examined (1951). Synge's method can however be used for the study of motions of four vortex filaments by making some simplifying assumptions. We study in particular the case where the configuration is initially a parallelogram and the vortices at the ends of diagonals are of equal strength.

By taking the adjacent sides and a diagonal which completely specify the configuration as the geometrical magnitudes, it is possible to classify all the motions in a manner similar to Synge's and the present paper deals with this problem. The theory resembles that of the motion of three vortices in many respects, but it is felt that it may be of interest to see the changes due to the presence of an additional vortex. Under the more particular case of rectangular configuration with special choice of vortex strengths, the motion is examined by representing the configurations by Cartesian co-ordinates.

2. EQUATIONS OF MOTION

Vortices of strengths $K_1$, $K_2$, $K_3$, $K_4$ lie initially at the corners $A$, $B$, $C$, $D$ of a parallelogram. Let $a$, $c$ and $b$, $g$ be the adjacent sides and diagonals respectively, the diagonal $b$ joining the vortices $K_1$, $K_1$ and $g$ joining...
K₁, K₂. Then these vortices remain at the corners of a parallelogram at the end of any time (Gorjatschaff, 1898). The configuration is uniquely determined by the magnitudes \( a, b, c \) except for orientation and the motion of the vortices is represented by a point on a representative plane by trilinear co-ordinates.

The following equations of motion of the vortices have been obtained by Lakshmana Rao (1951).

\[
\frac{da}{dt} = S a^{-1} [K_1 (b^2 - c^2) - K_2 (g^2 - c^2)] \\
\frac{db}{dt} = S b^{-1} [2K_2 (c^2 - a^2)] \\
\frac{dc}{dt} = S c^{-1} [K_1 (a^2 - b^2) - K_2 (a^2 - g^2)]
\]

where \( S = \) area of the parallelogram ABCD.

We have the integrals due to Kirchhoff (Lamb, 1945).

\[
2K_1K_2 \log \frac{ac}{b} + K_1^2 \log b + K_2^2 \log g = \text{const.} = a \tag{2.4}
\]
\[
K_1b^2 + K_2g^2 = \text{const.} = \beta. \tag{2.5}
\]

### 3. Variable Configurations

Following Synge's procedure we write

\[
x_1 = a / (a + b + c) \tag{3.1}
\]
\[
x_2 = b / (a + b + c) \tag{3.2}
\]
\[
x_3 = c / (a + b + c) \tag{3.3}
\]

and take \( x_1, x_2, x_3 \) to signify the trilinear co-ordinates of a point with an equilateral triangle of unit attitude for the triangle of reference so that

\[
x_1 + x_2 + x_3 = 1. \tag{3.4}
\]

For any parallelogram configuration we have definite values of \( a, b, c \) giving a unique representative point \( x_1, x_2, x_3 \). For a given \( x \)-point we have an infinity of similar parallelogram configurations. As in Synge's theory of three vortices, the possible collinear configurations correspond to points on the lines of join of the middle points of the reference triangle. Essentially there are two distinct collinear configurations only. Differentiation of the equations (3.1), (3.2), (3.3) gives

\[
\frac{dx_1}{dt} = \frac{da}{dt} (a + b + c)^{-1} - a (a + b + c)^{-2} \left( \frac{da}{dt} + \frac{db}{dt} + \frac{dc}{dt} \right)
\]
and two similar equations. It is seen that
\[ \frac{dx_1}{dt} = KG_1, \quad \frac{dx_2}{dt} = KG_2, \quad \frac{dx_3}{dt} = KG_3 \]
with
\[ K = S (a + b + c)^4 (abcg)^{-2}, \] (3.5)
\[ G_1 = -K_1x_1 (x_2^2 - x_3^2) x^2 + K_2x_1 (x^2 - x_3^2) x_2^2 + x_1 \theta \] (3.6)
\[ G_2 = -2K_2x_2 (x_3^2 - x_1^2) x^2 + x_2 \theta \] (3.7)
\[ G_3 = -K_1x_3 (x_1^2 - x_2^2) x^2 + K_2x_3 (x_1^2 - x_2^2) x_2^2 + x_3 \theta \] (3.8)
and
\[ \theta = K_1x_1 (x_2^2 - x_3^2) x^2 + 2K_2x_2 (x_3^2 - x_1^2) x^2 + K_1x_3 (x_1^2 - x_2^2) x^2 \]
\[ - K_2x_1 (x_2^2 - x_3^2) x_2^2 - K_2x_3 (x_1^2 - x_2^2) x_2^2, \] (3.9)
where
\[ g / (a + b + c) = x, \] (3.10)
so that
\[ x^2 = 2x_1^2 - x_2^2 + 2x_3^2. \] (3.11)
The differential equations
\[ \frac{dx_1}{G_1} = \frac{dx_2}{G_2} = \frac{dx_3}{G_3} = K dt. \] (3.12)
give the motion of the representative point. They define a congruence of
\[ x \]-curves giving the behaviour of the configurations and also the rate of
change while the size of the configuration is given by one of the integrals
(2.4), (2.5).

The following results are then simply deduced.

If the strengths of the vortices do not satisfy the relation
\[ K_1^2 + K_2^2 + 4K_1K_2 = 0 \] (3.13)
and \( a \) is known from the initial configuration (equation 2.4) to each \( x \)-point
there corresponds a unique parallelogram configuration except for orienta-
tion. Whereas if they satisfy the relation (3.13) then to each point on the
conic (hyperbola)
\[ K_1x_3^2 + K_2x_3^2 = 0 \] (3.14)
there corresponds a single infinity of parallelogram configurations of both
orientations, and to a point off this conic there corresponds a unique paral-
lelogram configuration except for orientation.
4. Singular Points

The equations

\[ G_1 = -K_1 x_1 (x_2^2 - x_3^2) x^2 + K_2 x_1 (x^2 - x_3^2) x_2^2 + x_1 \theta = 0 \]  
\[ G_2 = -2K_2 x_2 (x_3^2 - x_1^2) x^2 + x_2 \theta = 0 \]  
\[ G_3 = -K_1 x_3 (x_1^2 - x_2^2) x^2 + K_2 x_3 (x_1^2 - x^2) x_2^2 + x_3 \theta = 0 \]

give the singular points of the representation curves. Under the condition \( K_1 + K_2 = 0 \) the collisions of unequal vortices, \( \nu/z., \) the cases \( x_1 = 0, \) \( x_2 = x_3 = \frac{1}{2} \) and \( x_3 = 0, x_1 = x_2 = \frac{1}{2}, x_3 = 0 \) are seen to correspond to singular points.

To see any other singular points we first consider the case where \( \theta = 0. \) Then from (4.2) we see that \( x_1 = x_3 \) and additionally (4.1) or (4.3) gives

\[ K_1 (x_2^2 - x_1^2) x^2 - K_2 (x_1^2 - x_1^2) x_2^2 = 0 \]

The case \( x_1 = x_2 = x_3 \) is seen to be exceptional and is to be ruled out.

In the case of \( \theta \neq 0, \) we have \( x_1 \neq x_3. \) We have then the easily obtainable relation

\[ \theta (K_1^2 + 4K_1K_2 + K_2^2) = 2K_2 (x_3^2 - x_1^2) (K_2 x_2^2 + K_2 x_3^2) \]

In case equation (3.13) holds we are led to equation (3.14)

\[ K_1 x_2^2 + K_2 x_3^2 = 0. \]

We have thus the results:

If

\[ K_1^2 + K_2^2 + 4K_1K_2 \neq 0 \]

and

\[ K_1 + K_2 \neq 0 \]

the singular points arise when \( x_1 = x_3 \) corresponding to rhombus configurations. The case \( x_1 = x_2 = x_3 \) corresponding to an angle of 60° in the rhombus is an exceptional case.

If

\[ K_1^2 + K_2^2 + 4K_1K_2 = 0 \]

we necessarily have (4.7) and the singular points lie on the hyperbola

\[ K_1 x_2^2 + K_2 x_3^2 = 0. \]

If

\[ K_1 + K_2 = 0 \]

(4.6) is necessarily true and the only singular points are when \( x_1 = 0, \)
\( x_2 = x_3 = \frac{1}{2} \) or \( x_3 = 0, x_1 = x_2 = \frac{1}{2} \) corresponding to collisions in pairs of unequal vortices.
5. Rectangular Configuration

If vortices of strengths $K, -K, K, -K$ lie at the corners of a rectangle initially, they do so at the end of any time (Lamb, 1945) and the configuration can never pass through collinearity. The initial orientation is thus reserved for all time.

To study the motion we may write

$$a = x, \quad c = y \quad 0 < x, y < \infty$$

and take $(x, y)$ as the rectangular cartesian co-ordinates of a point on a representative plane. The co-ordinate axes are non-reachable barriers for the motion of $(x, y)$.

The equations of motion are easily seen from (2.1), (2.3) to be

$$-\frac{dx}{x^3} = \frac{dy}{y^3} = \frac{2K}{xy(x^2 + y^2)} \, dt.$$  

The representative point moves on the curve

$$x^{-2} + y^{-2} = \text{const.} = 1/d^2$$

which is equivalent to the integral (2.4).

This congruence of curves has no singularities and the configuration cannot remain fixed.

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REFERENCES