NOTE ON THE PERTURBATION CALCULATION
OF PHASE SHIFTS FOR CENTRAL AND
NON-CENTRAL INTERACTIONS*

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INTRODUCTION

It is well known that the mutual scattering of two particles is completely
determined by the asymptotic phase shifts in the appropriate
wave function.
The purpose of this note is to point out that the calculation of the phase
shifts can be reduced to an ordinary eigenvalue problem (for a closed state).
By applying the usual perturbation theory to the latter, one obtains readily
the phase shifts as series in powers of the interaction potential; the first
order term is, as is to be expected, identical with Born's approximation. The
procedure, it will be shown, is applicable to non-central as well as centrally
symmetric interactions.

2. CENTRALLY SYMMETRIC INTERACTIONS

Consider first scatterings due to centrally symmetric interactions. For
discussing the motion in the centre of gravity system, one can regard one
of the scattering partners as a fixed scattering centre and the other as the
scattered particle having the reduced mass $\mu$ of the system. Let us imagine
the latter particle enclosed in a large empty sphere (radius $R$) and work out
the shifts of its energy levels when the scattering centre is introduced at the
centre of the sphere. For a state with the orbital quantum number $l$, the
asymptotic radial wave functions with and without the scattering centre are
respectively:

$$\sim \frac{1}{r} \sin (kr - ln/2 + \eta), \quad (a)$$

$$\sim \frac{1}{r} \sin (kr - ln/2) \quad (b) \quad (1)$$

* This work was completed while the authors were at the Department of Theoretical
Physics, University of Liverpool.
† For an elegant discussion on the convergence of Born's expansion see Jost & Pais (1951).
where $\eta_l$ is the phase shift required for the scattering problem and $k$ is the wave number of the particle in the centre of mass system. Clearly the wave number $k$ must in either case be such that the wave function has a node at $r = R$ (the corresponding eigenvalue $= \hbar^2 k^2 / 2\mu$). Consequently the difference of $k$ in the two cases must be exactly so as to compensate for the appearance of the phase shift in (a), thus,

$$R \Delta k = - \eta_l$$

As $R \to \infty$, $\Delta k$ is infinitesimal, hence the shift in the eigenvalue is given by

$$\Delta E = \Delta \left( \frac{\hbar^2 k^2}{2\mu} \right) = \frac{\hbar^2 k}{\mu} \Delta k = - \frac{\hbar^2 k \eta_l}{R \mu}.$$

The phase shifts have thus been expressed in terms of eigenvalues of certain closed states.

The series expansion for $\eta_l$ follows directly, when one considers the interaction potential as a perturbation and applies the usual perturbation theory to determine the energy shift $\Delta E$. When properly normalized for sufficiently larger $R$, the unperturbed radial wave functions are given by

$$R_{\ell}^0(r) = \left( \frac{\pi \hbar}{R} \right)^{1/2} J_{\ell + \frac{1}{2}}(kr).$$

where $J_{\ell + \frac{1}{2}}(kr)$ are the half integral Bessel functions. Making use of (4) one obtains, for instance, the first and second order terms of $\eta_l$ explicitly as

$$\eta_l^{(1)} = - \left( \frac{\mu \pi \hbar^2}{h^2 R^2} \right) \int J_{\ell + \frac{1}{2}}(kr) V(r) r dr$$

$$\eta_l^{(2)} = \left( \frac{2 \pi \hbar^2}{h^2 R^2} \right) \sum_{k' + k} \frac{k'}{k^2 - k'^2} \left\{ \int J_{\ell + \frac{1}{2}}(kr) J_{\ell + \frac{1}{2}}(k'r) r dr \right\}^2$$

$$= \left( \frac{2 \pi \hbar^2}{h^2} \right) \int \frac{k' dk'}{k^2 - k'^2} \left\{ \int J_{\ell + \frac{1}{2}}(kr) V(r) J_{\ell + \frac{1}{2}}(k'r) r dr \right\} \times \left\{ \int J_{-\ell - \frac{1}{2}}(k'r) V(r') J_{\ell + \frac{1}{2}}(k'r) r' dr' \right\}$$

$$= (-1)^\ell \left( \frac{2 \pi \mu \hbar^2}{h^2} \right) \int J_{\ell + \frac{1}{2}}(kr) V(r) J_{\ell + \frac{1}{2}}(k'r) r dr \times$$

$$\times \int J_{\ell + \frac{1}{2}}(k'r) V(r') r' dr'.$$
where, for the large $R$ assumed, the summation over $k'$ has become an integral: for the latter, it is clear from the way it has been derived that Cauchy's principal value should be taken. The result of the corresponding integration, namely,
\[
\int_{\kappa'} J_{\kappa'}(k'r) J_{\kappa'}(k'r') k'dk' = \left( -1 \frac{\pi}{2} \right) J_{\kappa-1/2}(kr) J_{\kappa+1/2}(kr')
\]
for $r' \sim r$, is obtained with the help of a contour integral which is an easy modification of an example given in Watson's book (1944) on Bessel functions.

The first order expression, we observe, is identical with Born's approximation. Using relatively involved methods, similar expansions for the phase shifts have been obtained by Helland (1941) for central interactions. His second order expression (28) after correcting a slight misprint, in that the last Bessel function occurring in (28) should read $J_{\kappa+1/2}(kr)$ instead of $J_{\kappa-1/2}(kr)$ can be easily seen to be identical with our expression (6).

3. NON-CENTRAL INTERACTIONS

For non-central forces, we shall consider as an example the Schwinger type of tensor forces. Here a component with given total angular momentum $J$ contains 3 orbital waves $L = J$, $J \pm 1$, the radial solutions of the two latter being coupled together by equations of the form:
\[
\frac{d^2 u_{j-1}}{dr^2} + \left[ k^2 - w_1(r) - \frac{J(J-1)}{r^2} \right] u_{j-1} = w_2(r) u_{j+1}
\]
(8)
\[
\frac{d^2 u_{j+1}}{dr^2} + \left[ k^2 - w_3(r) - \frac{(J+1)(J+2)}{r^2} \right] u_{j+1} = w_2(r) u_{j-1}
\]
(9)
where $w_1(r)$, $w_2(r)$ and $w_3(r)$ are known functions of the radial factors of the central and non-central potentials and have the dimension of energy multiplied by $\mu\hbar^2$.

For the solution to represent the scattering of an incident plane wave, the $u$-functions must have the asymptotic forms:
\[
u_{j-1} \sim A_{j-1} \frac{1}{r} e^{i\eta_{j-1}} \sin [kr - (J-1)\pi/2 + \eta_{j-1}]
\]
(10)
\[
u_{j+1} \sim A_{j+1} \frac{1}{r} e^{i\eta_{j+1}} \sin [kr - (J+1)\pi/2 + \eta_{j+1}]
\]
where $A_{j-1}$, $A_{j+1}$ are certain known constants.

In order to apply the perturbation method, one formally reduces the coupled equations to a pair of independent homogeneous equations by
writing the right-hand-side expressions of (8) and (9) respectively as

\[
\left( w_2(r) \frac{\mu_{j+1}}{\mu_{j+1}} \right) u_{j+1} \quad \text{and} \quad \left( w_2 \frac{\mu_{j+1}}{\mu_{j+1}} \right) u_{j+1};
\]

thus \( u_{j-1}, u_{j+1} \) become virtually solutions for the radial potentials:

\[
-w_1(r) - w_2(r) \frac{\mu_{j+1}}{\mu_{j+1}} \quad \text{and} \quad -w_3(r) - w_4(r) \frac{\mu_{j-1}}{\mu_{j+1}}.
\]

Let us regard \( w_1(r), w_2(r) \) and \( w_3(r) \) as quantities of the first order for the perturbation calculation and make the corresponding series developments of \( u_{j-1} \) and \( u_{j+1} \). The above defined virtual potentials can be expressed as similar series containing terms of all orders. Although these potentials contain the unknown \( u \)-functions, for the perturbation calculation they are effectively as though given. For, to the \( n \)th order, the virtual potentials contain only \( u \)-functions to the \( n \)-1th order. Thus when the \( n \)th order terms in the potentials are required, they can already be calculated explicitly from the solutions of the previous stage. Consider, for instance, the stage of the calculation when the virtual potentials are known to the \( n \)th order. The phase shifts \( \eta_{j-1}^{(n)}, \eta_{j+1}^{(n)} \) of the \( n \)th order and the corresponding wave functions of the closed states can be determined with the perturbation method as in the case of the central forces. The proper \( u \)-functions to the same order are obtained by normalizing the wave functions in accordance with the requirements (10). From the \( n \)th order \( u \)-functions the \( n \)-1th order terms in the virtual potentials can be calculated, hence the perturbation calculation can proceed one stage further. Following this procedure the first order phase shifts can be written down as readily as in the case of central forces and are found to be identical with the corresponding expressions given by Ashkin and Wu (1948) using Born's approximation. The second order corrections can be described as follows:—

Let

\[
\eta_{(a)^{(2)}} = 2\pi^2 (-1)^a \left[ \int_0^\infty J_{(a+\frac{1}{2})} (kr') H_a(r') J_{-(a-\frac{1}{2})} (kr') r'dr' \times \right.
\]

\[
\left. \times \int_0^r J_{(a+\frac{1}{2})} (kr) H_a (r) rdr \right] + \frac{A_{\delta}}{A_a} e^{i\eta_{(a+1)} - \eta_{(a+1)}} \int_0^\infty \left( I_a J_{(a+\frac{1}{2})} (kr) - I_a J_{(a-\frac{1}{2})} (kr) \right) \left( r w_2 (r) dr \right), \quad (11)
\]

where

\[
H_a(r) = w_1(r) + \frac{A_{\delta}}{A_a} J_{(a+\frac{1}{2})} (kr) H_a (r) = w_2(r) + \frac{A_{\delta}}{A_a} J_{(a-\frac{1}{2})} (kr) \quad (12)
\]
and

\[ I_a = \frac{1}{2} \left[ J_{(a+\frac{1}{2})} (kr) \int_0^r H_\alpha (r') J_{(a+\frac{3}{2})} (kr') r'dr' + J_{(a+\frac{1}{2})} (kr) \times \right. \\
\left. \int_0^\infty H_\alpha (r') J_{(a+\frac{3}{2})} (kr') J_{(a+\frac{1}{2})} (kr') r'dr' \right] \]

(13)

\( I_b \) being the same as \( I_a \) with \( a \) everywhere replaced by \( b \). Then \( \eta_{J-1} \) and \( \eta_{J+1} \) are given by putting in (11) \( a = J - 1, b = J + 1 \) and \( a = J + 1, b = J - 1 \) respectively. The terms containing \( I_a \) and \( I_b \) are due to the first order corrections to the wave functions. It may be noticed here, that though the first order phase shifts (Born’s approximation) are real, the second order expressions are complex quantities, thus giving complex phase-shifts, as is actually the case (see for example, Ashkin and Wu, 1948).

As far as we know the expression for second order corrections to the phase-shifts for non-central interactions have not been given before.

**SUMMARY**

It is shown that the calculation of the phase-shifts can be reduced to an ordinary eigen-value problem (for a closed state). By applying the usual perturbation theory to the latter, one obtains readily the phase shifts as series in powers of the interaction potential. The procedure, it is shown, is applicable to non-central as well as centrally symmetric interactions. Explicit expressions for phase shifts up to second order are given for both kinds of potentials.

**REFERENCES**

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Jost, R. and Pais, A .. Ibid., 1951, 82, 840.