NOTE ON A PAIR OF FUNCTIONAL EQUATIONS SATISFIED BY $T_n(z)$

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Art. 1.—It is well known that Tschebyscheff's function $T_n(z)$ forms a particular solution of the functional equation:

$$f_{n+1}(z) - z f'_n(z) + \frac{1}{n} f_{n-1}(z) = 0, \quad (n > 1) \quad (I)$$

as also of the differential equation:

$$(1 - z^2)^{n/2} \frac{d^2w}{dz^2} - z \frac{dw}{dz} + n^2w = 0. \quad (A)$$

It is interesting to verify that $T_n(z)$ satisfies another functional equation, viz.,

$$(1 - z^2)f'_n(z) + n\{f_{n+1}(z) - \frac{1}{n} f_{n-1}(z)\} = 0, \quad (n \geq 1). \quad (II)$$

We shall write (I) and (II) in the equivalent forms,

$$\frac{1}{n} f_{n-1}(z) = zf_n(z) - \frac{1}{n} (z^2 - 1) f'_n(z) \quad (III)$$

and

$$2f_{n+1}(z) = zf_n(z) + \frac{1}{n} (z^2 - 1) f'_n(z). \quad (IV)$$

We now propose to investigate all possible common solutions of (I) and (II) or what is the same thing (III) and (IV). We shall henceforth refer to the differential equation (A) as the $T$-equation of rank $n$ and symbolise it as $T^{(n)}$.

To fix our ideas, suppose that for some fixed value of $n$, $f_n(z)$ is an arbitrary solution of the equation $T^{(n)}$, so that

$$(1 - z^2)f'_n(z) - zf_n(z) + n^2f_n(z) = 0. \quad (1)$$

Then introduce the two "contiguous" functions $f_{n-1}(z)$ and $f_{n+1}(z)$ in accordance with the relations (III) and (IV).

Differentiating (IV) and utilising (1), we find

$$2f_{n+1}'(z) = (n + 1) \left\{f_n(z) + \frac{z}{n} f'_n(z)\right\}. \quad (2)$$
A second differentiation, combined with (II), leads to:

\[ 2f_{n+1}''(z) = \left[ \frac{(n+1)^2}{n} + \frac{n+1}{n} \cdot \frac{z^2}{1-z^2} \right] f_n'(z) - \frac{n(n+1)z}{1-z^2} f_n(z). \]  

(3)

The relations (2) and (3) accord with the relation:

\[ (1-z^2)f_{n+1}''(z) - zf_{n+1}'(z) + (n+1)^2f_{n+1}(z) = 0, \]  

(4)

showing that \( f_{n+1}(z) \) satisfies \( T^{(n+1)} \).

Further, from (IV) and (2) we get:

\[ 2zf_{n+1}(z) - \frac{1}{n+1} \cdot (z^2-1)f_{n+1}'(z) \]

\[ = z \left\{ zf_n(z) + \frac{1}{n} (z^2-1)f_n'(z) \right\} - \frac{z^2-1}{n+1} \cdot (n+1) \left\{ f_n(z) + \frac{zf_n'(z)}{n} \right\} \]

\[ = f_n(z), \]

i.e.,

\[ \frac{1}{2}f_n(z) = zf_{n+1}(z) - \frac{1}{n+1} \cdot (z^2-1)f_{n+1}'(z). \]

(5)

This signifies that \( f_n(z) \) is one of the two functions "contiguous" to \( f_{n+1}(z) \).

If we now define the second "contiguous" function \( f_{n+2}(z) \) of \( f_{n+1}(z) \) by:

\[ 2f_{n+2}(z) = zf_{n+1}(z) + \frac{1}{n+1} \cdot (z^2-1)f_{n+1}'(z), \]

and recognise that the other "contiguous" function of \( f_{n+1}(z) \), viz., \( f_n(z) \), satisfies a relation, viz., (5), which is similar in form to the equation (III) for \( f_{n+1}(z) \), we infer on the strength of the results proved before that \( f_{n+2}(z) \) satisfies \( T^{(n+2)} \). Proceeding in this way, we derive an "ascending" sequence of functions, viz.,

\[ f_n(z), f_{n+1}(z), f_{n+2}(z), \]

(7)

which conform respectively to the sequence of differential equations:

\[ T^{(n)}, T^{(n+1)}, T^{(n+2)}, \]

and between any two contiguous members of which there subsist the two functional relations (I) and (II).

We shall now look for a "descending" sequence of functions, possessing similar properties.

For this purpose we may differentiate (III) and utilise (I), so as to deduce

\[ \frac{1}{2}f_{n+1}'(z) = (n-1) \cdot \left\{ \frac{z}{n} f_n'(z) - f_n(z) \right\}. \]

(8)
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Another differentiation combined with (1) leads to:

$$\frac{1}{2} f_{n-1}''(z) = \left[ \frac{(n-1)^2}{n} + \frac{n-1}{n} \cdot \frac{z^2}{1-z^2} \right] f_n'(z) - \frac{n(n-1)}{1-z^2} \cdot f_n(z). \quad (9)$$

Manifestly (III), (8) and (9) abide by the relation $T^{(n-1)}$, viz.,

$$\left(1 - z^2\right) f_{n-1}''(z) - z f_{n-1}'(z) + (n-1)^2 f_{n-1}(z) = 0. \quad (10)$$

Repetition of the previous line of argument leads to a "descending" sequence of functions:

$$f_n(z), f_{n-1}(z), f_{n-2}(z), \ldots, f_0(z), \quad (11)$$

satisfying the sequence of equations of the type $T^{(n)}$ and at the same time the pair of functional equations (I), (II) or (III), (IV).

It is scarcely necessary to add that, whereas the "ascending" sequence (7) comprises an infinite number of elements, the "descending" sequence (11) comprises only a finite number.

Art. 2.—It is now easy to make use of the results of the foregoing article to investigate the common solutions of the sequence of differential equations of the type $T^{(n)}$ and of the pair of functional equations (I) and (II). The needful for this is to take the initially fixed value of $n$, as contemplated in § 1, to be unity. That is to say, we have to start with an arbitrary solution $f_1(z)$ of $T^{(1)}$, so that

$$\left(1 - z^2\right) \cdot f_1''(z) - z f_1'(z) + f_1(z) = 0. \quad (12)$$

Application of the foregoing process at once places at our disposal a sequence of functions, beginning with $f_1(z)$, viz.,

$$f_1(z), f_2(z), \ldots, f_n(z), \ldots$$

which are compatible with (I) and (II) or (III) and (IV). As regards $f_0(z)$, which is one of the two functions, "contiguous" to $f_1(z)$, we have, on putting $n = 1$ in (III),

$$f_0(z) = 2 \left(1 - z^2\right) \cdot f_1'(z) + z f_1(z). \quad (13)$$

In as much as the left side of (12) is

$$\frac{d}{dz} \left\{ \left(1 - z^2\right) \cdot f_1'(z) + z f_1(z) \right\},$$

we get

$$\left(1 - z^2\right) \cdot f_1'(z) + z f_1(z) = \text{const. (say, c)}; \quad (14)$$

so that

$$f_0(z) = 2 \cdot c. \quad (15)$$
Thus finally we obtain the common solutions of the pair of functional equations (I) and (II), or (III) and (IV), viz.,

\[ f_0(z), f_1(z), f_2(z), \ldots, f_n(z), \ldots. \]

We may remark that the common solutions involve two and only two effective numerical parameters, these being the two arbitrary constants inherently present in the arbitrary solution \( f_1(z) \) of the differential equation of the second order \( T^{(2)} \). It is needless to point out that the constant value \( (2c) \) of \( f_0(z) \) is not an independent entity, but is rather dependent upon the two constants of \( f_1(z) \) by virtue of (14) and (15).

It will be observed that the common solutions of (I) and (II) automatically satisfy the sequence of differential equations \( \{T^{(n)}\}^* \).

In the succeeding Article we propose to scrutinise in some detail the relationship that subsists among (III), (IV) and \( T^{(n)} \).

Art. 3.—Starting with the two functional equations (III), (IV) and the differential equation \( T^{(n)} \),

we shall combine them in three distinct pairs, viz.,

(III) and (IV), (III) and (A), (IV) and (A)

and study the legitimate inferences that can be drawn in each case.

Case I.—First suppose that \( f_n(z) \) satisfies (III) and (IV), (see Art. 1).

Then from (IV),

\[
\frac{1}{2} \cdot \frac{d}{dz} \left \{ zf_n(z) + \frac{1}{n} \cdot (z^2 - 1) f'_n(z) \right \} = f_{n+1}(z)
\]

\[
= \frac{n+1}{z^2-1} \cdot \left \{ zf_{n+1}(z) - \frac{1}{2} f_n(z) \right \},
\]

from (III)

\[
= \frac{n+1}{z^2-1} \cdot \left [ z \cdot \frac{1}{2} \left \{ zf_n(z) + \frac{1}{n} \cdot (z^2 - 1) f'_n(z) \right \} - \frac{1}{2} f_n(z) \right ],
\]

from (IV)

which when simplified leads to (A).

* The differential equation \( T^{(0)} \) being of the degraded form:

\[
(1 - z^2)f''_0(z) - zf'_0(z) = 0,
\]

it is clear that \( f_0(z) = \text{const.} \) is one of its solutions.
CASE II.—Next suppose that \( f_n(z) \) satisfies (III) and (A). Then

\[
\frac{1}{2} \cdot \left\{ z f_{n-1}'(z) + \frac{z^2 - 1}{n-1} \cdot f_{n-1}(z) \right\}
\]

\[
= z \cdot \left\{ z f_n'(z) - \frac{1}{n} \cdot (z^2 - 1) f_n''(z) \right\}
\]

\[
+ \frac{z^2 - 1}{n-1} \cdot \left\{ z f_n'(z) + f_n(z) + \frac{1}{n} \cdot f_n''(z) - \frac{2z}{n} f_n'(z) \right\},
\]

\[
= f_n(z), \quad \text{on simplification by use of (A)}.
\]

Then from (III)

\[
\therefore \frac{z^2 - 1}{n-1} \cdot f_{n-1}'(z) = 2f_n(z),
\]

or,

\[
\frac{1}{n} \cdot (z^2 - 1) f_n'(z) = 2f_{n+1}(z),
\]

which is no other than (IV).

CASE III.—Now suppose that \( f_n(z) \) satisfies (IV) and (A). Then

\[
2 \cdot \left\{ z f_{n+1}'(z) - \frac{1}{n+1} \cdot (z^2 - 1) f_{n+1}(z) \right\}
\]

\[
= z \cdot \left\{ z f_n'(z) + \frac{1}{n} \cdot (z^2 - 1) f_n''(z) \right\}
\]

\[
- \frac{1}{n+1} \cdot (z^2 - 1) \left\{ z f_n'(z) + f_n(z) + \frac{1}{n} \cdot (z^2 - 1) f_n''(z)
\right.
\]

\[
\left. + \frac{2z}{n} f_n'(z) \right\},
\]

\[
= f_n(z), \quad \text{on simplification by aid of (A)}.
\]

\[
\therefore \frac{1}{n+1} \cdot (z^2 - 1) f_{n+1}'(z) = \frac{1}{2} f_n(z),
\]

or,

\[
\frac{1}{n} \cdot (z^2 - 1) f_n'(z) = \frac{1}{2} f_{n-1}(z),
\]

which is the same as (III).

Thus the three equations (III), (IV) and (A) are so related that the combination of any two of them automatically leads to the third.

Art. 4.—We shall close this topic with a brief reference to certain special solutions. It is directly verified that each of the three functions:

\[
f_n(z) = \frac{1}{2^{n-1}} \sin(n \cos^{-1}z), \quad \frac{1}{2^n} \cosh(n \cosh^{-1}z), \quad \frac{1}{2^{n-1}} \sinh(n \cosh^{-1}z)
\]

(16)
satisfies the two functional equations (I) and (II) and the differential equation (A). Besides, the function $V_n(z)$, defined by
\[
V_n(z) = \frac{1}{2^n} \cdot \frac{1}{z^n} \cdot F\left(\frac{n}{2}, \frac{n + 1}{2}, 1 + n; \frac{1}{z^2}\right), \quad (|z| > 1)
\]
and called "Tschebyscheff's Function of the Second Kind" is known to satisfy (I) and (A). So $V_n(z)$ must conform to the equation (II).

As remarked in the paper referred to in the foot-note, the general solution of (I), which is after all a functional equation of the second order, can be thrown into the form:
\[
f_n(z) = a_n(z) \cdot \theta(z) + \beta_n(z) \cdot \phi(z),
\]
where $a_n(z)$ and $\beta_n(z)$ are two particular solutions of (I) and $\theta(z)$, $\phi(z)$ are two arbitrary functions.

For obvious reasons, $a_n(z)$ and $\beta_n(z)$ can at pleasure be replaced by any two of the functions $T_n(z)$, $V_n(z)$ and those marked (16).

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* Refer to Bagchi and Chakravarty's paper, viz., "Note on Tschebyscheff function $T_n(z)$ and its associated equations," where certain properties of the "so-called" Tschebyscheff's function of the second kind are discussed. (Vide Journal of the Indian Math. Society, 1950.)

† Throughout this paper, the parameter $n$ is supposed to be a positive integer.