SOME SOLUTIONS OF THE WAVE EQUATION

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1. For rectilinear boundaries a number of solutions of the two-dimensional wave equation

$$\nabla^2\phi = c^2 \frac{\partial^2\phi}{\partial t^2} \tag{1}$$

are known, in which the boundary condition satisfied by ϕ is

$$\phi = 0, \tag{2.1}$$

or

$$\frac{\partial\phi}{\partial n} = 0, \tag{2.2}$$

∂n denoting a normal element drawn to the boundary. In the transverse vibrations of membranes the condition (2.1) amounts to a fixed edge, and in Hydro-dynamics of inviscid liquids (2.2) is to be satisfied over a fixed boundary. Some such solutions satisfying (2.1) have been given by B. R. Seth.[1] For hexagonal and equilateral boundaries* incomplete solutions have been obtained by D. G. Christopherson[2] and B. B. Sen.[3] Christopherson's solution has been completed by P. N. Sharma.[4]

2. When the boundary condition is nonhomogeneous solutions of the wave equation for rectilinear boundaries have not received much attention, though formal solutions can be obtained through the Green's function. In a recent investigation of some problems in the bending of elastic plates it was required to determine solutions of (1) satisfying the boundary condition

$$\phi = k, \text{ a constant.} \tag{2.3}$$

It is proposed to give some of these solutions. It is found that for equilateral, some rhombus and pentagonal boundaries the solution can be obtained in a finite number of trigonometrical terms.

3. Assuming that $\theta \propto \cos(pt/c)$ we can rewrite (1) as

$$\nabla^2\phi + p^2\phi = 0. \tag{3.1}$$

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* The complete solution for an equilateral boundary was originally obtained by Lamé in *Leçons sur l'Elasticite.*

A typical solution of (3·1) is

$$\phi = {}^{\sin}_{\cos}\,(p\sqrt{}\cos\theta)\,{}^{\sin}_{\cos}\,(py\sin\phi). \tag{3·2}$$

It may be noted that when applied to the vibrations of a membrane the condition (2.3) means that its edge is vibrating harmonically.

4. For a rectangular section with sides $x = \pm a$, $y = \pm b$, we see that $k\cos px\sec pa$ satisfies the boundary condition (2·3) over $x = \pm a$. Using the Fourier's expansion for $\cos px$ which vanishes for $x = \pm a$, we get

$$\phi = k\left[\cos px\sec pa - \sum^{\infty} A_n \sec b\sqrt{p^2 - \alpha_n^2}\cos\alpha_n x\cos\sqrt{p^2 - \alpha_n^2}\,y\right], \tag{4}$$

where $\alpha_n = (2n+1)\pi/2a$,

$$A_n = \frac{2(-1)^{n+1}kp^2}{a\alpha_n(p^2 - \alpha_n^2)}.$$

For a right-angled isosceles triangular section with sides $x = a$, $y = a$, $y + x = 0$, we see that the functions to be used for the individual sides can be taken to be of the form $\sin px$, $\sin py$ and $\cos\frac{p}{\sqrt{2}}(x+y)$, respectively.

On account of the symmetry which exists about the line $y = x$ we now take two Fourier series which are such that their sum vanishes over $y = -x$ and in which terms can be obtained from one another by interchanging x and y. Thus we get

$$\begin{aligned}\phi &= k\cos\frac{p}{\sqrt{2}}(x+y) + \tfrac{1}{2}k(1 - \cos pa\sqrt{2})\left[\frac{\sin px}{\sin pa} + \frac{\sin py}{\sin pa}\right]\\ &- \sum_{0}^{\infty} A_n \operatorname{cosec} a\sqrt{p^2 - \alpha_n^2}\left[\cos\alpha_n x\sin\sqrt{p^2 - \alpha_n^2}\,y + \cos\alpha_n y\sin\sqrt{p^2 - \alpha_n^2}\,x\right]\\ &- \sum_{0}^{\infty} B_n \sec a\sqrt{p^2 - \beta_n^2}\left[\sin\beta_n x\cos\sqrt{p^2 - \beta_n^2}\,y + \sin\alpha_n y\cos\sqrt{p^2 - \beta_n^2}\,x\right]\end{aligned} \tag{5}$$

where $\alpha_n = (2n+1)\pi/2a$, $\beta_n = n\pi/a$, and

$$A_n = \frac{(-1)^{n+1}kp^2\cos^2\frac{1}{2}pa\sqrt{2}}{a\,\alpha_n(\frac{1}{2}p^2 - \alpha_n^2)},\quad B_n = \frac{(-1)^{n+1}k\,\beta_n p^2\sin^2\frac{1}{2}pa\sqrt{2}}{a(\frac{1}{2}p^2 - \beta_n^2)(p^2 - \beta_n^2)}.$$

For an equilateral section with sides $y = a$, $y = \pm x\sqrt{3}$, we should take functions of the type $\cos py$, $\sin py$, $\cos\frac{1}{2}px\sqrt{3}\cos\frac{1}{2}py$ and $\cos\frac{1}{2}px\sqrt{3}\sin\frac{1}{2}py$. It is found that no Fourier expansions are required, and we get

$$\begin{aligned}\phi &= k\left[2\cos\tfrac{1}{2}py\cos\tfrac{1}{2}px\sqrt{3} - \cos py\right]\\ &\quad - k\cot\tfrac{1}{2}pa\left[2\sin\tfrac{1}{2}py\cos\tfrac{1}{2}px\sqrt{3} - \sin py\right].\end{aligned} \tag{6}$$

The result for a rhombus with sides $y = \pm x\sqrt{3} \pm 4\pi n/p$, n being an integer, can be readily deduced from (6) and we get

$$\phi = k\,[2\cos\tfrac{1}{2}py\cos\tfrac{1}{2}px\sqrt{3} - \cos py.] \qquad (7)$$

In like manner for a pentagonal section with sides $y = a$, $y = \pm x\sqrt{3} \pm 4\pi n/p$, the solution given in (6) holds good.

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