THE RESOLUTION OF THE CLIFFORD ALGEBRA (DIRAC ALGEBRA) WITH ANY NUMBER OF SYMBOLS AS THE DIRECT SUM OF MINIMAL LEFT IDEALS

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INTRODUCTION

In this note the Clifford-Dirac Algebra generated by $n$-symbols $e_1, e_2, \ldots, e_n$ satisfying the relations

$$e_re_t + e_te_r = 2\delta_{rt} \quad \text{(Kronecker symbol)}$$

over a ground field whose characteristic $= 2$ and which contains $\sqrt{-1}$ is resolved into the sum of minimal left ideals. These ideals as well as their bases have been chosen in a suitable manner and the corresponding representation is seen to be identical with the well-known one given by Weyl and Brauer.¹

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§1. It is well known² that the basis elements of the Clifford Algebra with the $n$ symbols $e_1, e_2, \ldots, e_n$ satisfying the relation

I. $e_re_t + e_te_r = 2\delta_{rt} \quad \text{(Kronecker symbol)}$ are all expressed succinctly by the expression $e_1^{\lambda_1}, e_2^{\lambda_2}, \ldots, e_n^{\lambda_n}$ where the $\lambda^r$ are integers mod 2. Evidently there are $2^n$ basis elements.

We deduce easily,

II. (1) $e_1e_2 \ldots e_{2r+1}$ commutes with the $e_p$; $p < 2r + 1$

and anticommutes with the $e_p$; $p > 2r + 1$.

II. (2) $e_1e_2 \ldots e_{2r}$ anticommutes with the $e_p$; $p < 2r$

and commutes with the $e_p$; $p > 2r$. 

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II. (3) \((e_1e_2\ldots e_n)^2 = (-1)^r\)

II. (4) \((e_1e_2\ldots e_{2r+1})^2 = (-1)^r\)

It follows also that the algebra \(C_{2n+1}\) (generated by an odd number of symbols) can be resolved into a direct sum of two simple algebras each of which is isomorphic with a \(C_{2n}\): i.e.,

\[ C_{2n+1} = C_{2n+1}\omega + C_{2n+1}(1 - \omega) \]

where

\[ \omega = \frac{1 + e_1e_2\ldots e_{2n+1}}{2} \quad \text{if } n \text{ is even} \]

\[ \omega = \frac{1 + i e_1e_2\ldots e_{2n+1}}{2} \quad \text{if } n \text{ is odd}. \]

§2. We now take up the complete resolution of a \(C_{2n}\) as a direct sum of minimal, mutually orthogonal left ideals. For this purpose we make use of a result due to Witt, viz., that a Clifford Algebra with \(2n\) symbols is the direct product of \(n\) such algebras with 2 symbols. Witt has shown that

\[ C_{2n} = (1, e_1, e_2) \times (1, ie_1e_2e_3, ie_1e_2e_4) \times (\ldots \ldots) \ldots \ldots \times (\ldots) \]

where each of the brackets represents an algebra generated by the symbols contained in it. Therefore, the idempotents generating minimal left ideals in \(C_{2n}\) are given by

\[ \omega_r = \frac{(1 + e_1)}{2} \cdot \frac{(1 + i e_1e_2\ldots e_{2n-1})}{2} \ldots \frac{(1 + i e_1e_2\ldots e_{2n-1})}{2} \ldots \ldots n \]

factors. Corresponding to the two signs \(\pm\) in each of the brackets there are evidently \(2^n\) such \(\omega_r\)'s. We now show that

\[ \omega_r^2 = \omega_r \quad \text{and} \quad \omega_r\omega_s = 0 \quad \text{if } r \neq s. \]

Now

\[ \omega_r^2 = \left\{ \frac{(1 + e_1)}{2} \ldots \frac{(1 + i e_1e_2\ldots e_{2n-1})}{2} \right\} \]

\[ \left\{ \frac{(1 + e_1)}{2} \ldots \frac{(1 + i e_1e_2\ldots e_{2n-1})}{2} \right\} \quad \text{n even} \]

\[ = \left(\frac{1 + e_1}{2}\right)^2 \ldots \left(\frac{1 + i e_1e_2\ldots e_{2n-1}}{2}\right)^2 \]

\[ = \left(\frac{1 + e_1}{2}\right)^2 \ldots \left(\frac{1 + i e_1e_2\ldots e_{2n-1}}{2}\right) = \omega_r, \]
To show that \( \omega_r \omega_s = 0 \) \((r \neq s)\) we first observe that in the expressions for \( \omega_r \) and \( \omega_s \), there is at least one bracket which appears with a change of sign in it. Calling this the \( p \)th bracket, if it is \( \left( 1 + i e_1 e_2 \ldots e_{2^{p-1}} \right) \) in \( \omega_r \), it will be \( \left( 1 - i e_1 e_2 \ldots e_{2^{p-1}} \right) \) in \( \omega_s \). Since the factors in the brackets commute with each other, we can bring the \( p \)th brackets together in \( \omega_r \omega_s \). But

\[
\frac{1 + i e_1 e_2 \ldots e_{2^{p-1}}}{2} \cdot \frac{1 - i e_1 e_2 \ldots e_{2^{p-1}}}{2} = 0
\]

Hence \( \omega_r \omega_s = 0 \) if \( r \neq s \).

We next prove that \( \sum_{r=1}^{2^n} \omega_r = 1 \).

**Proof.**—Let the result be true for \( n = m \)

\[
\sum_{r=1}^{2^m} \omega_r = \frac{2^m}{2} (1 \pm e_1) \ldots \left( 1 \pm i e_1 e_2 \ldots e_{2^{m-1}} \right) = 1 \text{ for } C_{2^m}
\]

Hence for \( C_{2^{m+2}} \)

\[
\sum_{1}^{2^{m+1}} \omega_r = \sum_{1}^{2^{m+1}} \left( \frac{1 \pm e_1}{2} \right) \ldots \left( \frac{1 \pm i e_1 \ldots e_{2^{m-1}}}{2} \right) \left( \frac{1 \pm e_2 \ldots e_{2^{m+1}}}{2} \right) \\
= \left( \sum_{1}^{2^m} \left( \frac{1 \pm e_1}{2} \right) \ldots \left( \frac{1 \pm i e_1 \ldots e_{2^{m-1}}}{2} \right) \right) \left( \frac{1 + e_2 \ldots e_{2^{m+1}}}{2} \right) \\
+ \left( \sum_{1}^{2^m} \left( \frac{1 \pm e_1}{2} \right) \ldots \left( \frac{1 \pm i e_1 \ldots e_{2^{m-1}}}{2} \right) \right) \left( \frac{1 - e_2 \ldots e_{2^{m+1}}}{2} \right)
\]

\[
= \frac{1 + e_1 e_2 \ldots e_{2^{m+1}}}{2} + \frac{1 - e_1 e_2 \ldots e_{2^{m+1}}}{2} = 1
\]

\( i.e., \) the result is true for \( n = m + 1 \). But for \( n = 1 \),

\[
\sum \omega_r = \frac{1 \pm e_1}{2} = 1 \text{ and hence it is true for all } n.
\]

We now proceed to deduce the irreducible representation of the algebra \( C_{2^n} \) by choosing a suitable basis of the minimal left ideal \( L \): generated by one of \( \omega_r \)'s, say
we first of all show that
\[ e_{2k+1} \omega = (-1)^k i e_{2k} \omega \quad k = 1, 2, \ldots, n - 1. \]

**Proof.** (i) Let \( k \) be even. From II. 1.

\[
\begin{align*}
e_{2k} \omega &= e_{2k} \left( \frac{1 + e_1}{2} \right) \cdots \left( \frac{1 + i e_1 \cdots e_{2k-1}}{2} \right) \cdots \\
&= \left( \frac{1 - e_1}{2} \right) \cdots \left( \frac{1 - i e_1 \cdots e_{2k-1}}{2} \right) \left( e_{2k} - e_1 \cdots e_{2k-1} e_{2k+1} \right) \\
&\quad \cdots \left( \frac{1 + i e_1 \cdots e_{2k-1}}{2} \right) \\
e_{2k+1} \omega &= \left( \frac{1 - e_1}{2} \right) \cdots \left( \frac{1 - i e_1 \cdots e_{2k-1}}{2} \right) \left( e_{2k+1} + e_1 \cdots e_{2k} \right) \\
&\quad \cdots \left( \frac{1 + i e_1 \cdots e_{2n-1}}{2} \right) \\
&= \frac{1 - e_1}{2} \cdots \frac{1 - i e_1 \cdots e_{2k-1}}{2} [e_1 \cdots e_{2k-1}] \times \\
&\quad \left( e_{2k} - e_1 \cdots e_{2k-1} e_{2k+1} \right) \cdots \left( 1 + i e_1 \cdots e_{2n-1} \right) \text{ using II. 4.} \\
&= \left( \frac{1 - e_1}{2} \right) \cdots \left( \frac{1 - i e_1 \cdots e_{2k-1}}{2} \right) \left( e_{2k} - e_1 \cdots e_{2k-1} e_{2k+1} \right) \\
&\quad \cdots \left( \frac{1 + i e_1 \cdots e_{2n-1}}{2} \right) \\
&= i e_{2k} \omega. \tag{a}
\end{align*}
\]

(ii) Let \( k \) be odd.

\[
\begin{align*}
e_{2k} \omega &= \left( \frac{1 - e_1}{2} \right) \cdots \left( \frac{1 - e_1 \cdots e_{2k-1}}{2} \right) \left( e_{2k} - i e_1 \cdots e_{2k-1} e_{2k+1} \right) \\
&\quad \cdots \left( \frac{1 + i e_1 \cdots e_{2n-1}}{2} \right)
\end{align*}
\]
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\[ e_{2k+1} \omega = \left( \frac{1 - e_1}{2} \right) \ldots \left( \frac{1 - e_1 \ldots e_{2k-1}}{2} \right) \left( \frac{e_{2k+1} + ie_1 \ldots e_k}{2} \right) \ldots \left( \frac{1 + ie_1 \ldots e_{2n-1}}{2} \right) \]

\[ = \left( \frac{1 - e_1}{2} \right) \ldots \left( \frac{1 - e_1 \ldots e_{2k-1}}{2} \right) [ie_1 \ldots e_{2k-1}] \times \left( \frac{e_{2k} - ie_1 \ldots e_{2k-1} e_{2k+1}}{2} \right) \ldots \left( \frac{1 + ie_1 \ldots e_{2n-1}}{2} \right) \]

\[ = \left( \frac{1 - e_1}{2} \right) \ldots \left( \frac{ie_1 \ldots e_{2k-1} - i}{2} \right) \left( \frac{e_{2k} - ie_1 \ldots e_{2k-1} e_{2k+1}}{2} \right) \ldots \left( \frac{1 + ie_1 \ldots e_{2n-1}}{2} \right) \]

\[ = -i e_{2k} \omega. \quad (b) \]

Combining (a) and (b) we get

\[ e_{2k+1} \omega = (-1)^k ie_{2k} \omega. \]

We thus see that all the symbols with odd suffixes can be expressed in terms of those with even suffixes only and that \( e_1 \omega = \omega \). We therefore take, as the basis elements of the minimal left ideal generated by \( \omega \) the \( 2^n \) terms occurring in

\[ e_2^{\lambda_n} e_3^{\lambda_{n-1}} \ldots e_2^{\lambda_2} e_1^{\lambda_1} \omega = a_r \omega \text{ where the } \lambda's \text{ are integers mod.2.} \]

The \( e's \) are written down, as above, with the suffixes, always in the descending order and \( \lambda_1, \lambda_2, \ldots, \lambda_n \) take the values 0, 1 in the dictionary order. Each \( a_r \) represents a particular combination of the \( e's \) and \( r \) takes \( 2^n \) values. We add a direct proof that the \( a_r \omega \) are linearly independent.

Proof.—We first show that if \( a \) represents some combination of the \( e's \), \( a\omega_r = \omega a \) where \( \omega_r \) and \( \omega_s \) are two different mutually orthogonal idempotents. Now an \( a \) is of the form

\[ a = e_{2l} e_{2m} \ldots e_{2s} e_{2t}, \text{ where } n \geq l > m \ldots > s > t \geq 1 \]

Hence \( a\omega_r = e_{2l} e_{2m} \ldots e_{2s} e_{2t} \omega_r \).

From II. 1, when \( e_{2l} \) is taken to the right of \( \omega_r \), one can see easily that there will be a change of sign in the first \( t \) brackets only. If now \( e_{2s} \) is brought
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To the right of $\omega_r$, the signs will be restored in the first $t$ brackets, but a change of sign occurs in the next $s - t$ brackets and the last $n - s$ brackets remain unaltered. We thus observe that when all the $e$'s are taken to the right of $\omega_r$, it would have changed over to a different orthogonal idempotent $\omega_r$,

$$i.e., \ a\omega_r = \omega_r\alpha \ (r \neq s)$$

Let now $\sum_{r=1}^{2^n} a (\sigma_r \omega) = 0$, i.e.,

$$a_1\omega_1 + a_2\omega_2 + \ldots + a_r\omega_r + \ldots + a_{2^n}\omega_{2^n} = 0$$

i.e., $a_1\omega_1\alpha_1 + a_2\omega_2\alpha_2 + \ldots + a_r\omega_r\alpha_r + \ldots + a_{2^n}\omega_{2^n}\alpha_{2^n} = 0$

Multiply by $\omega_{sr} (r = 1, 2, \ldots, 2^n)$ on the left. We obtain

$$a_r\omega_{sr}\alpha = a_r\omega_r\alpha_r = 0$$

i.e., $a_r = 0, (r = 1, 2, \ldots, 2^n)$ i.e., the $a_r\omega$ are linearly independent.

Choosing these as the basis elements of the left ideal $\mathbb{L}$: generated by $\omega$, we can obtain the matrices of the representation in terms of the Pauli matrices

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The matrices are easily seen to be

$$e_1 \to P_1 \times P_1 \times P_1 \times \ldots \times P_1 \times P_1 \times P_1 \quad n \text{ terms}$$

$$e_{2^k} \to P_1 \times P_1 \times \ldots \times P_2 \times P \times E \times E \times \ldots \times E \quad ,$$

$$e_{2^k+1} \to (-1)^k P_1 \times P_1 \times \ldots \times P_1 \times P_3 \times E \times \ldots \times E \quad ,$$

where $P_3$ and $P_3$ occur in the $k$th place from the right end in the corresponding expressions.

REFERENCES
