# DIFFRACTION OF LIGHT BY ULTRASONIC WAVES

## (Deduction of the Different Theories from the Generalised Theory of Raman and Nath)

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### 1. INTRODUCTION

THE phenomenon of diffraction of light by ultrasonic waves, discovered simultaneously by Debye and Sears[1] in America and Lucas and Biquard[2] in France has been explained by various authors, *e.g.*, Brillouin[3], Rytov,[4] Erwin David,[5] Extermann and Wannier,[6] Van Cittert[7], Raman and Nath.[8 9 10] They have based their theories on the fundamental wave equation, derived from Maxwell's equations, governing the propagation of light through a quasi-homogeneous medium.

Brillouin,[3] considering the case of slight obliquity, has confined his attention to the first order lines. The main assumptions made are (*a*) that the diffraction pattern contains only first order lines, (*b*) and that the central beam loses practically no part of the intensity due to diffraction.

Rytov,[4] assuming the constancy of the central beam in the above sense, has considered diffraction lines of higher orders in addition to the first order lines. It may be mentioned here that the theory fails if the diffraction pattern contains several lines.

Erwin David[5] has obtained similar results for the intensities of the first and second order lines, with the help of integrals.

The original elementary theory put forward by Raman and Nath[8] is based on the assumption that the diffraction is due to the corrugated form of the emerging wave-front arising from the density fluctuations in the medium and can be calculated by considering the phase changes only, ignoring the amplitude changes. This theory is valid only when the ultrasonic frequency is low. Generally, the diffraction pattern contains many lines under such conditions.

The three theories mentioned earlier are found to be successful only when the diffraction pattern contains lines of first and second order, whereas the elementary theory of Raman—Nath holds good when many diffraction lines are present in the pattern. Therefore, in order to cover a greater range of frequency Raman and Nath put forward a generalised theory. This could be expected to contain the first three theories as special cases. Raman and Nath have obtained a difference differential equation, giving the amplitude of any diffraction line. Nath[10] later solved this equation, and found a limited number of terms in the different expressions for intensities. These calculations have been pushed forward by the present author in another paper.[11] The theory, though complete and exhaustive, involves cumbersome calculations and the expressions obtained for the intensities are non-converging.

Extermann and Wannier[6] have solved the above-mentioned difference differential equation with the help of Hill's equation to obtain numerical results, whereas Van Cittert[7] has obtained the solution as a series involving Bessel's functions.

In order to avoid any confusion about the selection of the theory, it is better to confine ourselves to a generalised theory of which particular cases may be considered to suit different experimental conditions. As the generalised theory of Raman and Nath has proved to be most successful, an attempt has been made to derive the different theories from it. Nath[12] himself has obtained a closed expression for the intensity of the first order line, when the incidence is normal. In this paper, the intensities of the first and second order lines have been calculated and the results thus obtained have been compared with the old results given by Brillouin, Rytov and Erwin David. The validity of the results obtained by Rytov and Erwin David, has been discussed and the conditions imposed therein have been found out *mathematically*.

## 2. Evaluation of the Intensities of First and Second Order Lines

Three different cases will be discussed below. Only the diffraction lines on the side favouring reflection will be considered. For diffraction lines on the other side of the zero order, similar analysis may be employed.

*Case I.—Intensities of the First Order Lines.*

Assumptions:

(*a*) The central beam traverses the ultrasonic region without any loss or gain in the intensity.

(*b*) The diffraction pattern does not contain lines of an order higher than one.

Following Nath's[10] analysis, the amplitude of the $n$th order diffraction line, satisfies the difference differential equation.

$$2 \cdot \frac{d\psi_n}{d\xi} - \psi_{n-1} + \psi_{n+1} = c_n \psi_n \tag{1}$$

and the boundary conditions are

$$\left.\begin{array}{l} \psi_n(0) = 0 \text{ for } n \neq 0 \\ \psi_0(0) = 1 \end{array}\right\} \tag{1 a}$$

Putting $n = 1$ in (1), we get

$$2 \cdot \frac{d\psi_1}{d\xi} - \psi_0 + \psi_2 = c_1 \cdot \psi_1$$

where $\lambda$ = wave-length of light *in vacuo*

$\lambda^*$ = wave-length of sound in liquid

$\mu_0$ = mean refractive index of the liquid.

$\mu_1$ = Maximum variation in the refractive index of the liquid.

$Z$ = distance traversed by light beam through ultrasonic region.

$\phi$ = angle of incidence.

$$\xi = \frac{2\pi \cdot \mu_1 \cdot Z}{\lambda}.$$

$$\rho = \frac{\lambda^2}{\mu_0 \cdot \mu_1 \cdot \lambda^{*2}}$$

$$a = \frac{\mu_0 \cdot \lambda^*}{\lambda} \cdot \phi$$

$$c_n = i\rho\,(n^2 + 2na).$$

Now since the central beam is unaffected $\psi_0 = 1$ and as the 2nd order line is absent $\psi_2 = 0$.

Therefore $$2 \cdot \frac{d\psi_1}{a\xi} - 1 = c_1\psi_1$$

or $$2\frac{d\psi_1}{d\xi} - c_1\psi_1 = 1$$

which has for its solution

$$\psi_1 = \mathrm{A} \cdot \exp.(c_1 \cdot \xi/2) - \frac{1}{c_1}$$

Boundary condition $\psi_1 = 0$

when $\xi = 0$ gives

$$\psi_1 = \frac{1}{c_1} \cdot \{\exp.(c_1 \cdot \xi/2) - 1\} \tag{2}$$

or $$\psi_1 = \exp.(c_1 \cdot \xi/4)\{\exp.(c_1 \cdot \xi/4) - \exp.(-c_1 \cdot \xi/4)\}/c_1.$$

Putting down the value of $c_1$, we find

$$\psi_1 = \frac{\exp.\{i(1+2\alpha)\rho\xi/4\}}{i\rho(1+2\alpha)}[\exp.\{i(1+2\alpha)\rho\xi/4\} - \exp.\{-i(1+2\alpha)\rho\xi/4\}]$$

The intensity of the first order line is therefore given by $T_{+1}{}^2$, where

$$T_{+1} = \frac{\sin\{(1+2\alpha)\rho\xi/4\}}{\rho(\alpha+\frac{1}{2})} \tag{3}$$

Using the notations of Erwin David, we find

$$\frac{\rho\xi}{2} = \cdot \tfrac{1}{2} \cdot \frac{\lambda^2}{\mu_0 \cdot \mu_1 \cdot \lambda^{*2}} \cdot \frac{2\pi\mu_1 \cdot Z}{\lambda} = \frac{\pi\lambda Z}{\mu_0 \cdot \lambda^{*2}}.$$

$$= \frac{\pi \cdot \lambda' Z}{\lambda^{*2}} \text{ ($\lambda'$ being the wavelength of light in the liquid $\lambda' = \lambda/\mu_0$)}$$

$$= \bar{x}_0 \text{ (because Z may be taken as the length of the cell as well).}$$

$$\rho = \frac{\mu_0 \cdot \lambda'^2}{\mu_1 \cdot \lambda^{*2}} = \frac{1}{\delta}$$

Therefore $$T_{+1} = \frac{\delta \cdot \sin\{(\alpha+\frac{1}{2})\bar{x}_0\}}{(\alpha+\frac{1}{2})} \tag{3a}$$

This is the expression obtained by Erwin David for the intensity of the first order line.

Again putting the values of $\rho$ and $\xi$ we find (3) reduces to

$$T_{+1} = \cdot \frac{\mu_1 \cdot \pi \cdot L}{\mu_0 \cdot \lambda'} \cdot \frac{\sin\left\{\frac{\pi \cdot L}{2\lambda^*} \cdot \left(\frac{\lambda'}{\lambda^*} + 2\phi\right)\right\}}{\frac{\pi L}{2\lambda^*} \cdot \left(\frac{\lambda'}{\lambda^*} + 2\phi\right)} \tag{3b}$$

The same expression has been obtained by Brillouin and Rytov. Thus, it may be concluded that Brillouin's theory is a particular case of the more general theory given by Raman and Nath. By making the assumption in this general theory, that the central beam is unaffected due to diffraction, the results of Brillouin can be deduced.

*Case II.—Intensity of the Second Order Line.*

Assumptions:

(*a*) The intensity of the central beam, undergoes no change due to diffraction.

(*b*) The diffraction lines of an order higher than two are not present in the pattern.

(*c*) The value of $\psi_1$, evaluated previously under the condition of the absence of the second order line, may be employed for further development.

The differential equation involving $\psi_2$ is therefore

$$2 \cdot \frac{d\psi_2}{d\xi} - \psi_1 = c_2 \cdot \psi_2 \qquad \text{(Since } \psi_0 = 1;\ \psi_3 = 0) \quad (4)$$

Hence

$$2 \cdot \frac{d\psi_2}{d\xi} - c_2\psi_2 = \frac{\exp.(c_1\xi/2) - 1}{c_1}$$

This equation has for its solution

$$\psi_2 = A^*{}_1 \exp.(c_2 \cdot \xi/2) + \frac{1}{c_1c_2} - \frac{\exp.(c_1\xi/2)}{c_1(c_2 - c_1)}$$

Since $\psi_2 = 0$ when $\xi = 0$.

$$A_1^* = \frac{1}{c_2(c_2 - c_1)}.$$

Therefore

$$\psi_2 = \frac{\exp.(c_2 \cdot \xi/2)}{c_2(c_2 - c_1)} - \frac{\exp.(c_1 \cdot \xi/2)}{c_1 \cdot (c_2 - c_1)} + \frac{1}{c_1c_2} \quad (5)$$

Putting down the values of $c_1$ and $c_2$

$$\psi_2 = -\frac{1}{\rho^2} \cdot \left[\frac{\exp.\{i(2 + 2a)\rho\xi\}}{4(1 + a)(3 + 2a)} - \frac{\exp.\{i(a + \frac{1}{2})\rho\xi\}}{(1 + 2a)(3 + 2a)} + \frac{1}{4(1 + a)(1 + 2a)}\right] \quad (6)$$

Using Rytov notations, we find

$$\rho\xi = 2u$$

$$\rho = h^{-1}$$

$$\alpha = \gamma/2.$$

$$\psi_2 = -h^2 \cdot \left[\frac{\exp.\{i(4+2\gamma)u\}}{(4+2\gamma)(3+\gamma)} - \frac{\exp.\{i(1+\gamma)u\}}{(3+\gamma)(1+\gamma)} + \frac{1}{(1+\gamma)(4+2\gamma)}\right] \tag{6a}$$

$$= -h^2 \cdot B_{+2} \text{ (in the notation of Rytov).}$$

The intensity of the second order line is therefore given by $T_{+2}^2$ where

$$T_{+2} = |h^2 \cdot B_{+2}|$$

Hence the result obtained by Rytov is also a special case of the generalised theory of Raman and Nath.

Writing down the expressions in full

$$I_{+2} = \frac{1}{\rho^4} \cdot \left[\frac{1}{16(1+\alpha)^2 \cdot (3+2\alpha)^2} + \frac{1}{(1+2\alpha)^2 \cdot (3+2\alpha)^2} + \frac{1}{16(1+\alpha)^2 \cdot (1+2\alpha)^2} + \frac{2\cos\{(2\alpha+2)\rho\xi\}}{16(1+\alpha)^2 \cdot (3+2\alpha)(1+2\alpha)} - \frac{2 \cdot \cos\{(\alpha+\frac{1}{2})\rho\xi\}}{4(1+2\alpha)^2 \cdot (3+2\alpha)(1+\alpha)} - \frac{2\cos \cdot \{(\alpha+3/2)\rho\xi\}}{4(1+2\alpha) \cdot (3+2\alpha)^2(1+\alpha)}\right]$$

$$= \frac{1}{16\rho^4 \cdot (1+\alpha)^2 \cdot (3+2\alpha)^2 (1+2\alpha)^2}.$$

$$[(1+2\alpha)^2 + 16(1+\alpha)^2 + (3+2\alpha)^2 + 2(3+2\alpha)(1+2\alpha)\cos\{(2\alpha+2)\rho\xi\} - 8(3+2\alpha)(1+\alpha)\cos\{(\alpha+\tfrac{1}{2})\rho\xi\} - 8(1+\alpha)(1+2\alpha)\cos\{(\alpha+\tfrac{3}{2})\rho\varsigma\}].$$

$$= \frac{1}{256 \cdot \rho^4 \cdot K} \Big[ (\alpha + 1)^2 \cdot [24 + 8 \cos \{(2\alpha + 2)\rho\xi\} - 16 \cdot \cos \{(\alpha + \tfrac{1}{2})\rho\xi\} - 16 \cdot \cos \{(\alpha + \tfrac{3}{2})\rho\xi\}] - 8 \cdot (\alpha + 1)[\cos \{(\alpha + \tfrac{1}{2})\rho\xi\} - \cos \{(\alpha + \tfrac{3}{2})\rho\xi\}] + 2 \cdot [1 - \cos \{(2\alpha + 2)\rho\xi\}] \Big]$$

where $K = (\alpha + 1)^2 \cdot (\alpha + \frac{3}{2})^2 \cdot (\alpha + \frac{1}{2})^2$.

Therefore

$$I_{+2} = \frac{1}{256\, \rho^4 K} \Big[ (\alpha + 1)^2 \cdot [16 + 16 \cos^2 \{(\alpha + 1)\rho\xi\} - 32 \cos \{(\alpha + 1)\rho\xi\} \cos \{\rho\xi/2\}] - 16(\alpha + 1) \sin \{(\alpha + 1)\rho\xi\} \sin \{\rho\xi/2\} + 4 \sin^2 \cdot \{(\alpha + 1)\rho\xi\} \Big]$$

$$= \frac{1}{4 \cdot \rho^4 \cdot k} \Big( [(\alpha + 1) \sin \{(\alpha + \tfrac{1}{2}) \cdot \rho\xi/2\} \sin \{(\alpha + \tfrac{3}{2})\rho\xi/2\}]^2 + \tfrac{1}{4} \cdot [\tfrac{1}{2} \cdot \sin \{(\alpha + 1)\rho\xi\} - (\alpha + 1) \sin (\rho\xi/2)]^2 \Big)$$

Using the notations of Erwin David, this may be written as

$$I_{+2} = \frac{\delta^4}{K} \Big( \tfrac{1}{4} \cdot (\bar{a} + 1) \sin \{(\bar{a} + \tfrac{1}{2}) x_0\} \sin \{(\bar{a} + \tfrac{3}{2}) \bar{x}_0\}]^2 + \tfrac{1}{16} [\tfrac{1}{2} \sin \{(\bar{a} + 1) 2\bar{x}_0\} - \sin \bar{x}_0\}]^2 \cdot \Big) \qquad (6b)$$

This expression is identical with that obtained by Erwin David. Therefore, the results of Erwin David and of Rytov, for the second-order lines, are nothing but special cases of the generalised theory of Raman-Nath under the conditions mentioned in the beginning.

*Case III.—Intensities of the first and second order lines.*

Assumptions:

(*a*) The intensity of the central beam is unity.

(*b*) Only the lines of first and second order are present in the diffraction pattern.

The differential equations involving $\psi_1$ and $\psi_2$ are

$$2 \cdot \frac{d\psi_1}{d\xi} - 1 + \psi_2 = c_1\psi_1 \tag{7}$$

$$2 \cdot \frac{d\psi_2}{d\xi} - \psi_1 = c_2 \cdot \psi_2 \tag{8}$$

Here the boundary conditions at $\xi = 0$, are

$$\psi_1 = \psi_2 = 0$$

$$\frac{d\psi_1}{d\xi} = \tfrac{1}{2}; \quad \frac{d\psi_2}{d\xi} = 0.$$

These equations can be solved simultaneously giving the values of $\psi_1$ and $\psi_2$.

Eliminating $\psi_1$, we have

$$4 \cdot \frac{d^2 \cdot \psi_2}{d\xi^2} - 2(c_1 + c_2)\frac{d\psi_2}{d\xi} + (1 + c_1c_2)\psi_2 = 1 \tag{9}$$

The auxiliary equation giving the complementary function is

$$4m^2 - 2m \cdot (c_1 + c_2) + (1 + c_1c_2) = 0.$$

Therefore

$$m = \frac{(c_1 + c_2) \pm \sqrt{(c_1 - c_2)^2 - 4}}{4}$$

Now, if 4 be negligible as compared to

$$(c_1 - c_2)^2 \tag{11}$$

we find

$$m = \frac{c_1}{2} \text{ or } \frac{c_2}{2}$$

The particular integral $= \dfrac{1}{1 + c_1c_2}$

As 1 is negligible compared to $c_1c_2$ due to (11), we have

$$\psi_2 = A^* \exp.(c_1\xi/2) + B^* \exp.(c_2\xi/2) + \frac{1}{c_1c_2}$$

The boundary conditions $\psi_2 = 0 = \frac{d\psi_2}{d\xi}$ when $\xi = 0$ give

$$\psi_2 = \frac{\exp.(c_2 \cdot \xi/2)}{c_2(c_2 - c_1)} - \frac{\exp.(c_1 \cdot \xi/2)}{c_1(c_2 - c_1)} + \frac{1}{c_1 \cdot c_2} \tag{12}$$

We find that (12) is identical with (5). Therefore, the restriction (*c*) in Case II, does not affect the expression for intensity of the 2nd order if condition (11) is applicable.

Since $(c_1 - c_2)^2 = -\rho^2 \cdot (3 + 2a)^2$, we find that condition (11) always holds good if $\rho$ is large, *i.e.*, either the sound intensity is low or the ultrasonic frequency is high. Under such conditions, the expression for the intensity of the second order line, as given by Rytov and Erwin David is always valid.

In order to find the intensity of the first order line, when the second order is also present in the diffraction pattern, we eliminate $\psi_2$ in equations (7) and (8) and find that is given by

$$4 \cdot \frac{d^2 \cdot \psi_1}{d\xi^2} - 2(c_1 + c_2)\frac{d\psi_1}{d\xi} + (1 + c_1 \cdot c_2)\psi_1 = -c_2 \tag{13}$$

This equation has for its solution

$$\psi_1 = \text{M}.\exp.(c_1\xi/2) + \text{N}\exp.(c_2\xi/2) - \frac{c_2}{1 + c_1c_2}$$

neglecting 1 as compared to $c_1c_2$ and applying the boundary conditions

$$\frac{d\psi_1}{d\xi} = \tfrac{1}{2} \text{ and } \psi_1 = 0, \text{ when } \xi = 0, \text{ we find}$$

$$\psi_1 = \frac{1}{c_1}\{\exp.(c_1\xi/2) - 1\} \tag{14}$$

Thus, we find that (14) is identical with (2). Hence, it follows that the results obtained by Erwin David and Rytov for the intensities of the 1st and 2nd order lines are true if we accept condition (11).

## 3. Discussion

It has been shown that the theories put forward by Brillouin, by Rytov and by Erwin David are special cases of the generalised theory of Raman and Nath. Since a band of research workers have proved the validity of the results of these authors, it can be definitely said that the Raman-Nath generalised theory holds good even when the diffraction pattern contains lines of lower orders only, which occurs either when the ultrasonic frequency is high or the sound intensity is low.

For ultrasonic waves having comparatively low frequency $c_n \to 0$ the difference differential equation (1) reduces to

$$2 \cdot \frac{d\psi_n}{d\xi} - \psi_{n-1} + \psi_{n+1} = 0$$

which has for its solution $\psi_n = J_n(\xi)$, provided $\psi_n(0) = 0$, $n \neq 0$ and $\psi_n(0) = 1$. Therefore, at low frequencies, the generalised theory reduces to the elementary theory.[8] Further, since validity of this theory at low frequencies has been well established, it follows that the generalised theory is applicable at low frequency as well. Thus it has been proved that the generalised theory holds good at all frequencies and the other theories are particular cases of this 'parent' theory.

## Conclusion

The closed expressions for the intensities of the first and second order lines have been calculated, on the basis of the generalised theory of Raman and Nath. The results are found identical with those obtained by Brillouin, Rytov and Erwin David. The conditions under which these results hold good have been discussed mathematically. The validity of the generalised theory at different frequencies has also been discussed.

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