ON A BESSEL FUNCTION OF THE SECOND KIND
AND WILKS' Z-DISTRIBUTION

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Wilks studied the distribution of \( Z = B \theta_2 \theta_3 \ldots \theta_n \), where \( \theta_i \) has the distribution

\[
dF = \frac{1}{\Gamma(a_i)} \theta_i^{a_i-1} e^{-\theta_i} d\theta_i; \quad (i = 1, 2, \ldots, n)
\]

He succeeded in integrating the distribution of \( Z \) for \( n = 2 \) and \( a_1 = \frac{1}{2} (N-1) \) and \( a_2 = \frac{1}{2} (N-2) \). Later Chung Tsi Hsu derived the distribution of \( Z \) for \( n = 2 \) and for any values \( a_1 \) and \( a_2 \) in connection with certain tests of hypotheses for samples from two Bivariate Normal populations. The distribution is derived in the form

\[
dF = \frac{2\sqrt{\pi} Z^{a_2-1} e^{-Z(B)} dZ}{B^{a_2} \Gamma(a_2) \Gamma(a_2-1) \Gamma(a_2-a_1+a_2+\frac{1}{2})} \times
\]

\[
\int e^{-2x} \frac{2\sqrt{Z/B} + x}{2\sqrt{Z/B} + x} x^{a_1-a_2+\frac{1}{2}} dx
\]

(1)

Taking \( B = 1, a_1 = a_2 = \frac{1}{2} (N - 1) \), this becomes

\[
dF = \frac{2\sqrt{\pi} Z^{a_2-1} e^{-Z(B)} dZ}{\Gamma(a_2) \Gamma(a_2-1/2)} \times
\]

\[
\int e^{-2x} \frac{2\sqrt{Z} + x}{2\sqrt{Z} + x} x^{-1/2} dx
\]

(2)

He says "Since (2) can apparently not be simplified, I have been unable thus far to find in manageable form the distribution of the ratio \( \frac{Z_1}{Z_2} \) and therefore of \( \frac{\mu'}{\mu} \). So he used the alternative criterion \( \omega = Z - Z' \) (or \( Z_1 - Z_2 \)) for the hypothesis \( H_1 \) that he considered.

2. In this paper I have derived the distribution of \( \frac{Z_1}{Z_2} \) in the most general case by the use of Bessel Function of the second kind and the particular case follows immediately.

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Now (1) can be written, changing the variable to \( t = 2\sqrt{Z/B} \), and
\[ 2x = t, \]
and
\[ dF = \frac{1}{2^{a_1+a_2-2}} \Gamma (a_1) \Gamma (a_2) \int^t_{a_1-a_2} k_{a_1+a_2-1} (t) dt \] (3)
where \( k_{a_1+a_2-1} (t) \) is a Bessel Function of the second kind.

Thus we have to consider a distribution of the form
\[ dF = y_0 x^m k_m (x) \] (4)
The distribution given by
\[ dF = y_0 e^{-cx/b} \left\{ \frac{\pi I_1 (x/b)}{\Gamma (m - t)} \right\} dx \] (5)
and
\[ dF = y_0 x^m k_m (x) \] (6)
have been studied by A. T. McKay\(^3\) and Karl Pearson\(^5\) respectively. So far I am not aware of the distribution (4) being studied.

3. Moments of the Distribution (4).—The constant \( y_0 \) is given by
\[ 1 = y_0 \int_0^\infty x^m k_m (x) \] (7)
\[ = y_0 2^{m-1} \Gamma \left( \frac{m + 1 + n}{2} \right) \Gamma \left( \frac{m + 1 - n}{2} \right) \ldots \quad \text{[cf.}]\]

or
\[ y_0 = 1/2^{m-1} \Gamma \left( \frac{m + 1 + n}{2} \right) \Gamma \left( \frac{m + 1 - n}{2} \right) \]

The moment generating function is
\[ M(t) = y_0 \int_0^\infty e^{tx} x^m k_m (x) \] (8)
Hence $\mu_r$—the $r$th moment about the origin is given by

$$
\mu_r = y_0 2^{m+r-1} \left[ \frac{(m + r + 1 + n)}{2} \right] r \left[ \frac{(m + r + 1 - n)}{2} \right]
$$

Substituting for $y_0$

$$
\mu_r = 2^r \frac{\Gamma \left( \frac{m + r + 1 + n}{2} \right) r \left( \frac{m + r + 1 - n}{2} \right)}{\Gamma \left( \frac{m + n + 1}{2} \right) \Gamma \left( \frac{m - n + 1}{2} \right)} \quad (9)
$$

From (9) we get the recurrence formula

$$
\mu_{2k+1}' = 2^2 \left( \frac{m + n + k}{2} \right) \left( \frac{m - n + k}{2} \right) \mu_{2k+1}' \quad (10)
$$

and

$$
\mu_{2k+2}' = 2^2 \left( \frac{m + n + 1 + k}{2} \right) \left( \frac{m - n + 1 + k}{2} \right) \mu_{2k}' \quad (11)
$$

4. Distribution of $v = \frac{x_1}{x_2}$ where $x_1$ and $x_2$ are distributed according to

$$
f(x_1) \, dx_2 = y_0 x_1^m k_{n_1}(x_1) \, dx_1
$$

$$
\phi(x_2) \, dx_2 = y_0 x_2^m k_{n_2}(x_2) \, dx_2 \quad (12)
$$

The distribution of $v$ is given by

$$
dF = dv \int_0^\infty y_0 y_0^1 v^{m_1 x_1^m} x_2^m k_{n_1}(v x_1) x_2^m k_{n_2}(x_2) \, dx_2
$$

$$
= y_0 y_0^1 v^{m_1} dv \int_0^\infty x_2^{m_1 + m_2} k_{n_1}(v x_1) k_{n_2}(x_2) \, dx_2 \quad (13)
$$

It is shown that the value of the integral is

$$
= 2^{m_1 + m_2} v^{m_1} \Gamma(\lambda) \Gamma(\mu) \int_0^\infty x_2^\mu \, dx_2
$$

where

$$
\lambda = \frac{m_1 + m_2 + n_1 - n_2}{2} + 1
$$

$$
\mu = \frac{1}{2} (m_1 + m_2 - n_1 + n_2) + 1
$$

$$
\nu = (m_1 + m_2) - (n_1 + n_2) + 1
$$

By putting $x_2 = \tan \theta$, the integral in (14) can be expanded in powers of $1 - v^2$ and we find the value of the integral to be

$$
= \frac{\Gamma \left( \frac{\nu + 1}{2} \right) \Gamma \left( \frac{k + 1}{2} \right)}{\Gamma \left( \frac{\nu + k}{2} + 1 \right)} \quad (15)
$$
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where
\[ k = 2\mu + 2\lambda - v - 2 \]
\[ \alpha = 1 - v^2 \]

and \( F \) is the Hypergeometric Function.

Substituting from (14) and (15) in (13), we get
\[
dF = y_0 y_0' y_0'' \frac{\Gamma(\lambda) \Gamma(\mu) \Gamma\left(\frac{\nu + 1}{2}\right) \Gamma\left(\frac{k + 1}{2}\right)}{\Gamma\left(\frac{\nu + k + 1}{2}\right)} \times v^{m+m} F\left(\frac{k + 1}{2}, \lambda, \frac{\nu + k + 1}{2} + 1, 1 - v^2\right) dv
\]
(16)

where \( y_0 = 1/2^{m-1} \Gamma\left(m_1 + \frac{1}{2} + n_1\right) \Gamma\left(m_1 + 1 - n_1\right) \)
and
\( y_0' = 1/2^{m-1} \Gamma\left(m_1 + \frac{1}{2} + n_2\right) \Gamma\left(m_1 + 1 - n_2\right) \)

(16) is the required distribution of \( x_1/x_2 \).

5. Particular Cases.—Put \( m = a_1 + a_2 - 1 \)
\[ n = a_1 - a_2 \]

We have for the distribution (3) the \( r \)th moment about the origin, given by
\[
\mu_r' = 2^r \frac{\Gamma\left(a_1 + \frac{r}{2}\right) \Gamma\left(a_2 + \frac{r}{2}\right)}{\Gamma(a_1) \Gamma(a_2)}
\]
(17)

The distribution of \( v = \frac{t_1}{t_2} \) is given by, taking \( m_1 = m_2 = a_1 + a_2 - 1 \), \( n_1 = n_2 = a_1 - a_2 \) and substituting in (16)
\[
dF = \frac{2B(2a_1, 2a_2)}{B^2(a_1, a_2)} v^{n_1-1} F\left(2a_1, a_1 + a_2, 2a_1 + 2a_2, 1 - v^2\right) dv
\]
(18)

where \( B \) is the Beta-Function.

In particular put \( a_1 = a_2 = \frac{1}{2} (N - 1) \) and the constant \( B \) (in \( t = 2 (\sqrt{Z/B}) = 1 \).

Then \( v = \frac{t_1}{t_2} = \frac{Z_1}{Z_2} \) is distributed as
\[
dF = \frac{2B(2a_1, 2a_1)}{B^2(a_1, a_1)} v^{n_1-1} F\left(2a_1, a_1 + a_2, 2a_1, 1 - v^2\right) dv
\]
(19)

Substituting \( v \) for \( v^2 \), we get the distribution of \( \frac{Z_1}{Z_2} \) as
\[
dF = \frac{B(2a_1, 2a_1)}{B^2(a_1, a_1)} v^{n_1-1} F\left(2a_1, a_1 + a_2, 4a_1, 1 - v\right) dv
\]
(20)

\( \Lambda^2 \)
which is the distribution required to test the hypothesis $H_1$, where $Z_1$ and $Z_2$ are distributed according to (2).

REFERENCES

5. .. "Further applications in statistics of $T_m (\lambda)$ Bessel Function," *ibid.*, 1932, 24, 293-350.