FINITE LONGITUDINAL VIBRATIONS

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Received December 10, 1946

It is known that, if we neglect terms of the second order, the differential equation satisfied by the displacement ξ in the case of longitudinal vibrations of strings and rods is

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}. \tag{1}$$

It is also known that, assuming Hooke's law holds good for large values of ξ, this equation is not merely an approximation, but is accurate for large values of ξ. But Hooke's law does not hold good for large values of ξ. In such a case the theory of Finite Strain gives the relation between the tension T and the ordinary stretch s as[1]

$$\mathrm{T} = \tfrac{1}{2}\mathrm{E}\left[1 - \frac{1}{(1+s)^2}\right], \tag{2}$$

E being Young's modulus. It will be therefore of interest to determine what form the differential equation satisfied by ξ takes for finite values of ξ. It is found that in such cases the equation is exactly the same as the general equation for long waves in a uniform canal with vertical sides.

Let A be a fixed point of the string AB when unstretched and placed in a straight line. Let PQ be any element of the unstretched string, P′Q′ the same element at the time t. Let AP $= x$, AP′ $= x'$. Let T and T $+ d$T be the tensions at P′ and Q′, and let m be the mass of a unit length of the unstretched string. The equation of motion then is

$$m \frac{\partial^2 x'}{\partial t^2} = \frac{\partial \mathrm{T}}{\partial x}. \tag{3}$$

As $s = \frac{\partial}{\partial x}(x' - x)$,

$$\mathrm{T} = \tfrac{1}{2}\mathrm{E}\left[1 - \left(\frac{\partial x'}{\partial x}\right)^{-2}\right], \quad \frac{\partial \mathrm{T}}{\partial x} = \mathrm{E}\frac{\partial^2 x'}{\partial x^2}\left(\frac{\partial x'}{\partial x}\right)^{-3} \tag{4}$$

(3) now takes the form

$$\frac{\partial^2 x'}{\partial t^2} = \frac{\mathrm{E}}{m} \cdot \frac{\partial^2 x'}{\partial x^2}\left(\frac{\partial x'}{\partial x}\right)^{-3}. \tag{5}$$

If the unstretched state of the string be chosen as the standard of reference, we can put $x' = x + \xi$, and we get

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{E}{m} \frac{\dfrac{\partial^2 \xi}{\partial x^2}}{\left(1 + \dfrac{\partial \xi}{\partial x}\right)^3}, \qquad (6)$$

which is of the same form as the general equation for long waves.[2]

Proceeding, as is usual in such cases, we find that the complete integral of (5) is[3]

$$x' = \alpha x + \beta t + c_1, \qquad (7)$$

where the arbitrary constants α and β are connected by the relation

$$\beta = \pm 2c(1 - \alpha^{-\frac{1}{2}}), \quad c^2 = E/m.$$

The general integral is

$$\left.\begin{aligned} x' &= \alpha x \mp 2c(1 - \alpha^{-\frac{1}{2}})\, t + \phi(\alpha) \\ o &= x \mp c\alpha^{-3/2}\, t + \phi'(\alpha) \end{aligned}\right\} \qquad (8)$$

Putting $u = \partial x'/\partial t$, we get

$$u = 2c(\alpha^{-\frac{1}{2}} - 1), \qquad (9)$$

and

$$x' - t(\tfrac{3}{2}u + c) = \phi(\alpha) - \alpha\phi'(\alpha), \qquad (10)$$

ϕ being an arbitrary function.

From (10) we get

$$u = f[x' - t(\tfrac{3}{2}u + c)], \qquad (11)$$

f being an arbitrary function. Thus, in such cases, as is well known, waves cannot be propagated without change of form.

Summary

The theory of Finite Strain gives the interesting result that the differential equation for finite longitudinal vibrations of strings and rods is the same as the general equation for long waves.

References

1. Seth, B. R. .. *Phil. Trans. Roy. Soc.*, 1935, **234**, 231.
2. Lamb, H. .. *Hydrodynamics*, 5th ed., 1930, 241.
3. ——— .. *Loc. cit.*, 458.