A CONGRUENCE PROPERTY OF $\tau(n)$

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Ramanathan and, more recently, Bambah and Chowla have proved by different methods involving the use of certain relations between Ramanujan's functions $P$, $Q$ and $R$, that

$$\tau(n) \equiv n \sigma_3(n) \pmod{7} \tag{1}$$

where $\tau(n)$ is defined by the relation

$$\sum_{r=1}^{\infty} \tau(n)x^r = x[(1-x)(1-x^2)(1-x^3)\ldots]^2, \quad |x| < 1; \tag{2}$$

and

$$\sigma_3(n) = \sum_{d|n} d^3. \tag{3}$$

I give below a proof which is independent of such relations. All congruences are modulo 7.

We have

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\left(\prod_{r=1}^{\infty} (1-x^r)^3\right)^3$$

$$= x \left(\prod_{r=1}^{\infty} (1-x^r)^3\right)^3$$

$$= x \left(\sum_{s=0}^{\infty} (-1)^s \frac{(2u+1)(2v+1)}{2} \right) \left(\sum_{s=0}^{\infty} (-1)^s (2v+1) \frac{(2v+1)^2}{2}\right).$$

Hence

$$\tau(n) = \sum_{s=0}^{\infty} (-1)^s (2u+1)(2v+1)$$

where $u, v$ run through the non-negative solutions of the equation

$$n = 1 + \frac{7u(u+1)}{2} + \frac{v(v+1)}{2}$$

which can be put in the form

$$8n = 7(2u+1)^2 + (2v+1)^2. \tag{5}$$

If $\binom{n}{7} = -1$, (5) has no solution. Therefore

$$\tau(n) = 0 \text{ when } n^3 = -1. \tag{6}$$

\[\text{End of Document}\]
Also if \( n = 0 \), then from (5)
\[
2v + 1 = 0.
\]
Hence
\[
\tau(7m) = 0. \quad (7)
\]
Now consider the case when \( n \) is equal to an odd prime \( p \) other than 7, such that
\[
\left(\frac{p}{7}\right) = 1.
\]
Then the equation
\[
p = x^2 + 7y^2 \quad (8)
\]
has a unique solution in positive integers \( x, y \) of opposite parity. If \((x_1, y_1)\) be this solution, then
\[
8p = (x_1 + 7y_1)^2 + 7(x_1 - y_1)^2 = (x_1 - 7y_1)^2 + 7(x_1 + y_1)^2 \quad (9)
\]
provides the two solutions of (5), giving
\[
\tau(p) = 2x_1^2 = 2p = p(p^3 + 1). \quad (10)
\]
In view of the relations (6), (7) and (10), we have
\[
\tau(p) = p\sigma_3(p) \quad (11)
\]
for all primes \( p > 2 \). It holds also when \( p = 2 \) because
\[
\tau(2) = -24 = 2\sigma_3(2).
\]
Using Mordell's identity
\[
\tau(p^\lambda) = \tau(p)\tau(p^{\lambda-1}) = p^{\lambda^2} \tau(p^{\lambda-2}), \quad \lambda \geq 2.
\]
it is now easily shown that
\[
\tau(p^\lambda) = p^{\lambda}\sigma_3(p^\lambda).
\]
Since \( \tau(n) \) and \( \sigma_3(n) \) are both multiplicative functions
\[
\tau(n) = n\sigma_3(n).
\]