UMBILICAL PROJECTION*

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Received February 8, 1944

(Communicated by Prof. B. S. Madhava Rao, F.A.SC.)

The purpose of the paper is to illustrate the use of Umbilical Projection and establish a few theorems relating to certain sets of circles and families of coaxal circles in a plane by making use of a number of known results in space.

We start with a quadric Q in space and note the following correspondence:

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* Read at the meeting of the Mathematical Society of the Panjab University, held on the 16th March 1942, in continuation of the paper published by the author in the Proc. Ind. Acad. Sci., Bangalore, 1942, 15A, No. 1.
Space S.  

(ix) **Polar lines for Q.**  ..  **Conjugate families** of coaxal circles.

(x) **Conjugate lines for Q.**  ..  Two families will be said to be 'polar' when the conjugate of one intersects the other.  *The relation is mutual.*

(xi) **A self-polar tetrahedron** for Q.  Four mutually orthogonal circles will be said to form a 'self-orthogonal tetrad,' for the family determined by two of them is conjugate to that of the other two.

(xii) **A self-conjugate pentad** for Q.  Five circles, related in the manner that the circle orthogonal to three of them belongs to the family determined by the other two, will be said to form a 'self-orthogonal pentad'.

(xiii) **A self-conjugate hexad** for Q.  Six circles, related that the circle orthogonal to three of them and the other three belong to a linear congruence will be said to form a 'self-orthogonal hexad'.

(xiv) **Moebius Tetrads**  ..  Two tetrads of circles ABCD and A'B'C'D, will be said to be Moebius if a circle of one tetrad, say A', and three of the other tetrad, say B, C, D, belong to a linear congruence. *The relation between the two is mutual.*

(xv) **Reciprocal Tetrahedra** for Q.  Two tetrads of circles ABCD and A'B'C'D' will be said to be reciprocal if a circle of one, say A', is orthogonal to three of the other, say B, C, D and A, A'; ... will be said to be corresponding circles while BC, B'C'; ... Corresponding families.

(xvi) **A set of eight associated planes.**  The eight circles common to three independent quadric congruences will be said to form a set of associated circles.

(xvii) **A hexagon**  ..  Six circles in general give rise to 15 families that arrange into 60 'hexads'
Space S. of the nature that every given circle belongs to two families of a hexad. Two families of a hexad will be said to be opposite if the four circles determining them belong one to each of the other four families of the hexad.

(xviii) Double-six of lines Two sets of six families each will be said to form a double-six if five of one set are intersected by a family of the other set.

(xix) Opposite edges of a tetrahedron Given a tetrad of circles, they determine six families, two of them will be said to be opposite if they include all the four circles. There are three pairs of opposite families.

Now we are in a position to state the following theorems on circles in a plane without giving the details of the corresponding propositions, in space, that are hinted in brackets wherever necessary at the end of the theorem.

1. (i) A linear congruence of circles is determined by three circles in the manner that all the circles that are orthogonal to the circle orthogonal to the given three belong to it.

   (ii) A family and a circle also determine a linear congruence.

   (iii) Two families belonging to a linear congruence intersect.

   (iv) Two linear congruences have a family common.

   (v) Three linear congruences have a circle common.

2. If $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$, $A_4B_4C_4D_4$, $A_5B_5C_5D_5$, $A_6B_6C_6D_6$ be tetrads of circles belonging to seven linear congruences respectively, $A_1$, $B_1$, $C_1$, $D_1$ also belong to a linear congruence (Moebius tetrads).

3. If the families determined by pairs of circles $AA'$, $BB'$, $CC'$ have a circle common, the circles common to the pairs of families $BC$, $B'C'$, $CA$, $C'A'$, $AB$, $A'B'$ are coaxal and conversely. The theorem holds even if the six circles belong to a linear congruence (Desargue's theorem).

4. Let $A'$, $B'$, $C'$ be the circles of the families conjugate to those determined by the pairs of circles $QR$, $RP$, $PQ$ common with the linear...
congruence determined by the circles A, B, C such that the families AA', BB', CC' have a circle common, then, if P', Q', R' be the circles of the families conjugate to BC, CA, AB, common with the linear congruence PQR, the families PP', QQ', RR' have a circle common (B3*, Ex. 8, p. 42).

5. (i) Two circles, one from each of two given families can be found to be coaxal with a given circle.

(ii) If OHD₁, O₁KB be circles of two families of coaxals, the three pairs of circles L₁, M; N, C; A, P of pairs of families OK, O₁H; HB, KD₁; D₁O₁, B coaxal with a given circle D belong to a linear congruence. Let B₁ be a circle of this congruence, then the three pairs of circles L₁, M₁; A₁, P₁; N₁, C₁ of the above families coaxal with B₁ belong to another linear congruence that will contain D. The tetrads ABCD and A₁B₁C₁D₁ obtained here are Moebius (B3, Ex. 10, p. 63).

(iii) If the given families intersect, then the three pairs of above families also intersect and the circles common to the pairs are coaxal (Pappus' theorem).

6. (i) Given three families of coaxal circles, there are ∞ families intersecting them forming a system. The aggregate of the circles of the two systems of families form a quadric congruence such that each circle belongs to two families one from each system.

(ii) The circles common to a quadric congruence and a linear congruence form a conic series.

(iii) A linear congruence containing a family of a quadric congruence contains another also.

(iv) Two quadric congruences can have two conic series, a family and a cubic series, two intersecting families and a conic series or four families common such that the two conic series have two circles common; the cubic series and the family also have two circles common.

7. (i) There are two families of coaxal circles intersecting four general families of coaxals.

(ii) Take five arbitrary families of coaxal circles. By omitting each in turn we obtain five sets of four families each; and these four families have got two families intersecting them; we again have two circles one from each of these two families coaxal with a given circle; the ten circles so obtained belong to a linear congruence (B3, Ex. 14, p. 66).

* B3 refers to H. F. Baker, Principles of Geometry, Vol. 3, here as well as in what follows.
(iii) If the five families intersect another family, then each set of four families has one other family intersecting them, the five families so obtained also intersect a new family. We thus obtain two sets of six families each forming a double-six (B3, p. 159).

8. If the pairs of opposite families of a hexad determined by six given circles be intersecting, the six families will belong to a quadric congruence in which case we shall have six circles, one from each family, belonging to a conic series common to this congruence and a linear congruence. These six circles will also give rise to 60 hexads of families. With the property that the circles common to the pairs of opposite families of a hexad are coaxal (B3, Dandelin's figure of six generators of a quadric forming a skew hexagon leading to Pascal's theorem for hexagon inscribed in a conic, p. 45).

9. Given a tetrad of circles, we shall have two circles from each pair of opposite families of the tetrad coaxal with a given circle giving rise to six circles, similarly we have six other circles corresponding to another given circle. The twelve circles so obtained belong to a quadric congruence. (B3, Ex. 17, p. 54.)

10. If circles be drawn coaxal with a family of a given tetrad of circles passing through the points of intersection of the circles of the opposite family, the twelve circles so obtained belong to a quadric congruence. (B3, Ex. 16, p. 54.)

11. If PP'; QQ' be pairs of orthogonal circles and f, f' be the respective conjugate families of those determined by P, P' and Q, Q', then f, f', P, P', Q, Q' belong to a quadric congruence (B3, Ex. 13, p. 52).

12. (i) If ABCD, A'B'C'D' be two reciprocal tetrads of circles, the families determined by the pairs of corresponding circles AA', BB', CC', DD' belong to a quadric congruence called $\phi$ (ABCD) (B3, Ex. 7, p. 41).

(ii) Consider the twelve points of intersection of the six pairs of circles of the tetrad ABCD. Three of these lying on A, B, C define a circle. We may thus, in thirty-two ways, specify four circles each of which contains three of the points, no two of these circles intersect in one of the twelve points. If we consider the family determined by D' and the circle through the three points chosen on A, B, C and the three families determined by A', B', C' taken respectively with the corresponding circles, the four families so obtained belong to a quadric congruence (B3, Ex. 15, Pp. 53).

(i) *Six circles in general determine the series.* If seven circles of the series be common with a quadric congruence, the whole series belongs to the congruence.

(ii) We can have two series belonging to a quadric congruence having five given circles common.

(iii) Three independent, quadric congruences can be constructed to contain the six circles. A, B, C, A', B', C' and containing the pairs of families determined by B' C, BC'; CA', C'A; AB', A'B respectively, then these quadric congruences have further a family common which has got two circles common with the series determined by the above six circles.

(iv) Given five circles and a family of circles, we can construct a series which contains the given circles and has two circles common with the family; but we cannot in general construct a series to contain four given circles and having two circles common with each of two given families unless the circles and the families belong to a quadric congruence; further a series can be constructed containing three circles, and having two circles common with each of three given families; again four families can have two circles common with a series containing two circles given.

(v) A unique cubic series exists having two circles common with each of five given families in general and containing a given circle; infinite number of such series exist if the families are intersected by some other family.

(vi) There exist ten families that will have two circles each common with both of two given cubic series, hence it can be shown that there exist six cubic series having each two circles common with every one of six given families in general (B3, pp. 141-42).


(i) The quadric congruences having seven circles common have an eighth common.

(ii) The circles of two self-orthogonal tetrads and those of Moebius tetrads form two sets, hence Moebius tetrads can consist of two self-orthogonal tetrads in a special case.

(iii) The cubic series determined by six circles of the set will have two circles common with the family determined by the remaining circles of the set.

(iv) If the circles of the set be denoted by 1, 2, 3, 4, 5, 6, 7, 8, the family common to the congruences (linear) determined by 123 and 567 intersects the family determined by the circles common to the family 34.
and the congruence 678 and the circle common to 45 and 812; hence it follows that four families common to the pairs of congruences 123, 567, 234, 678, 345, 781, 456, 812 belong to a quadric congruence.

(v) Let 2, 3, 4, 5, 7, 8 be any six circles and 1 a further circle; let P, P'; Q, Q'; R, R' be the pairs of circles from the pairs of families 23, 57, 78, 34, 45, 82, coaxal with 1. The families QR', RP', PQ' and Q'R, R'P, P'Q belong to a quadric congruence. Let P_1, Q_1, R_1, P'_1, Q'_1, R'_1 be the other circles, common with this congruence, of the families 23, 34, 45, 57, 78, 82 respectively, then the families P_1, P'_1 Q_1Q'_1, R_1R'_1 have a circle common say 6 completing the set.

15. (i) If \( x_i = 0 \) (i = 1, 2, 3) be equations† of three independent quadric congruences, \( \theta^2 x_1 + 2\theta x_2 + x_3 = 0 \) represent a system of quadric congruences out of which there are eight of a particular type* that are determined by a circle and a conic series, the eight circles determining these eight congruences form a set of associated circles (Math. Student, June 1942, X, Q. 1809).

(ii) If three quadric congruences of circles have a family common they have four more circles common, if they have a conic series common, they have two more circles common (B3, Ex, 4, p. 154).

16. (i) If two pairs of opposite families of a tetrad of circles be conjugate the remaining families are also conjugate and the tetrad is self-orthogonal.

(ii) If two pairs of opposite families of a tetrad of circles be polar, the remaining families are also polar, similarly behave the reciprocal tetrad.

(iii) The families determined by the corresponding pairs of circles here have a new circle common forming with either of the tetrads a self-orthogonal pentad. The pairs of corresponding families are intersecting, giving six more circles that are orthogonal to the new circle obtained above. We have thus got 15 circles in all such that each one is orthogonal to six of them that are coaxal by threes forming four families. In fact, there are 15 circles, 20 families constituting 10 pairs of conjugate as well as polar families, and 15 linear congruences with the property that each circle belongs to four families and six congruences, each family contains three circles and belongs to three congruences, each congruence contains six circles and four families. The 15 centres of the circles are collinear by threes in 20 lines, four through each centre, forming 15 quadrilaterals whose vertices are the centres, each

* Coolidge, loc. cit., pp. 161 and 165, Th. 54.
vertex is common to six quadrilaterals which have a line common by threes. (B3, Ex. 5, p. 35).

17. As a special case we can construct a self-orthogonal pentad consisting of three point-circles and two proper circles: Let AB'CA'BC' be a hexagon such that every pair of adjacent sides are circular lines; if D, E, F be the meet of the opposite sides, the circles DEF, ABC and the point-circles A', B', C' or the circles DEF, A'B'C' and the point circles A, B, C form the required pentad. (B3, Ex. 9, p. 44.)

18. If φ (ABCE) be a quadric congruence constructed in the manner of § 12 where E is a circle of the congruence φ (ABCD), the two congruences have a family common, with D' as a member and φ (ABCE) contains the circle D. If F be the circle of φ (ABCE) common to the family having E as a member, intersecting the families AA', BB', CC', other than E, the six circles ABCDEF form a self-orthogonal hexad. In particular, if the circles D', E, F are coaxal, the circles A B C E F form a self-orthogonal pentad (B3, Ex. 10, p. 47).

19. A self-orthogonal tetrad or pentad taken with any other circle form a self-orthogonal pentad or hexad of circles respectively.

20. If the circles of either a self-orthogonal tetrad, pentad or hexad belong to a quadric congruence or a cubic series, an infinite number of such sets of circles can be found to belong to the congruence or the series. (B3, Ex. 12, p. 50; Ex. 27, p. 145).

21. Let a, b, c be the linear congruences of circles orthogonal to the circles A, B, C respectively, and p, q, r be the families of coaxals common to the pairs of congruences b, c; c, a; a, b respectively; then the congruences Ap, Bq, Cr have a family common.

22. If a variable family of coaxals moves about so that three fixed circles of it belong to three fixed linear congruences respectively, any fourth fixed circle of the family will then generate a quadric congruence. (Salmon, Vol. I, p. 118, Ex. 14).

23. If the variable families of a hexad contain each a fixed circle and five circles of the hexad belong each to a linear congruence, the sixth circle of the hexad then generates a cubic series (Salmon, Vol. I, Ex. 5, p. 145).