A NOTE ON THE σ-SYMBOLS

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As is well known (see Van der Waerden, 1932) the symbols $\sigma^{k}_{\lambda\dot{\mu}}$ are used to set up a connection between tensors and spinors for transformations of the Lorentz group. k is a tensor index running from 0 to 3 while λ and μ are spinor indices which can take the values 1 and 2 only.* Hitherto it has been usual to prescribe the numbers $\sigma^{k}_{\lambda\dot{\mu}}$ explicitly and to show that they remain unaltered when subjected to a Lorentz transformation and the associated spinor transformation simultaneously. In fact the spinor transformation associated to a given Lorentz transformation is, in effect, defined by this condition. In the present paper no use will be made of an explicit representation of the σ's. All their properties will be deduced from the defining equations (1) and (2). Besides compactness, this procedure has the advantage that the same equations and all their consequences remain valid even when the most general transformations not included in the Lorentz group are admitted. They can therefore be directly taken over to the general theory of relativity (*cf.* Infeld and Van der Waerden, 1933).

For the present the space-time is assumed flat and the metric tensor is taken to be $g_{kl} = 0$ $k \neq l$, $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$. Similarly the antisymmetric spinors $\epsilon^{\lambda\mu}$, $\epsilon^{\dot{\lambda}\dot{\mu}}$, $\epsilon_{\lambda\mu}$, $\epsilon_{\dot{\lambda}\dot{\mu}}$ used for raising and lowering the spinor indices are given by $\epsilon^{12} = \epsilon_{12} = 1$, $\epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{1}\dot{2}} = 1$. For any spinor a_{μ}

$$a^{\lambda} = \epsilon^{\lambda\mu} a_{\mu}, \; a_{\lambda} = a^{\mu} \epsilon_{\mu\lambda}$$

with similar relations for the dotted spinors. The σ's are *defined* by the two following conditions:

$$\overline{\sigma^{k}_{\mu\dot{\lambda}}} = \sigma^{k}_{\lambda\dot{\mu}} \tag{1}$$

$$\sigma^{\lambda\dot{\mu}}_{k} \, \sigma_{l\dot{\mu}\nu} = \delta^{\lambda}_{\nu} \, g_{kl} - \frac{i}{2} \epsilon_{klmn} \, \sigma^{m\lambda\dot{\mu}} \, \sigma^{n}_{\dot{\mu}\nu} \tag{2 a}$$

* In this paper Latin alphabets shall always denote tensor indices the Greek alphabets being reserved for spinor indices.

Here the bar denotes conjugate-complex and ϵ_{klmn} is a tensor antisymmetric in all the four indices with $\epsilon_{0123} = -1$. From (1) and (2*a*) it follows that

$$\sigma^{\lambda\dot\mu}_{k}\,\sigma_{l,\,\mu\dot\nu}{}^{\cdot} = \delta^{\dot\lambda}_{\cdot\dot\nu}\, g_{kl} + \frac{i}{2}\,\epsilon_{klmn}\,\sigma^{m\lambda\dot\mu}\,\sigma^{n}_{\mu\dot\nu}{}^{\cdot} \tag{2b}$$

The fact that the usual representation of σ's satisfies (2) becomes obvious when, in conformity with the usual method, one regards σ^0 as the two-rowed unit matrix and σ^1, σ^2, σ^3 as the three Pauli matrices and compares their commutation rules with (2). Some results, which are already well known (Infeld and Van der Waerden, 1933; Fierz and Pauli, 1939) follow immediately from (2)

$$\sigma^{k}_{\lambda\dot\mu}\,\sigma^{l\dot\mu\nu} + \sigma^{l}_{\lambda\dot\mu}\,\sigma^{k\dot\mu\nu} = 2\delta^{\nu}_{\lambda}\, g^{kl} \tag{3 a}$$

$$\sigma^{k}_{\lambda\mu}{}^{\cdot}\,\sigma^{l\mu\dot\nu} + \sigma^{l}_{\lambda\mu}{}^{\cdot}\,\sigma^{k\mu\dot\nu} = 2\delta^{\dot\nu}_{\dot\lambda}\, g^{kl} \tag{3 b}$$

$$\sigma^{\lambda\dot\mu}_{k}\,\sigma^{l}_{\mu\dot\nu}{}^{\cdot} = 2\delta^{l}_{k} \tag{3 c}$$

Also

$$\sigma^{k}_{\lambda\dot\mu}\,\sigma^{l\dot\mu\nu} - \sigma^{l}_{\lambda\dot\mu}\,\sigma^{k\dot\mu\nu} = -i\,\epsilon^{klmn}\,\sigma_{m\lambda\dot\mu}\,\sigma_{n}^{\dot\mu\nu} \tag{4 a}$$

$$\sigma^{k}_{\lambda\mu}{}^{\cdot}\,\sigma^{l\mu\dot\nu} - \sigma^{l}_{\lambda\mu}{}^{\cdot}\,\sigma^{k\mu\dot\nu} = i\,\epsilon^{klmn}\,\sigma_{m\lambda\mu}{}^{\cdot}\,\sigma_{n}^{\mu\dot\nu} \tag{4 b}$$

From the irreducibility of the Pauli matrices it follows that (Fierz and Pauli, 1939)

$$\sigma^{\dot\lambda\mu}_{k}\,\sigma^{k}_{\nu\dot\rho}{}^{\cdot} = 2\delta^{\dot\lambda}_{\cdot\dot\rho}\,\delta^{\mu}_{\nu} \tag{5}$$

However a direct proof based on (2) can be given as follows. Consider $\sigma^{k}_{\lambda\mu}{}^{\cdot}\,\sigma^{l\mu\dot\nu}\,\sigma^{m}_{\nu\rho}{}^{\cdot}$ Notice that from (2)

$$\begin{aligned}
\epsilon_{klmn}\,\sigma^{k}_{\mu\lambda}{}^{\cdot}\,\sigma^{l\dot\lambda\rho}\,\sigma^{m}_{\rho\nu}{}^{\cdot} &= -\frac{i}{2}\,\epsilon_{klmn}\,\epsilon^{klpq}\,\sigma_{p,\,\mu\lambda}{}^{\cdot}\,\sigma_{q}^{\dot\lambda\rho}\,\sigma^{m}_{\rho\nu}{}^{\cdot} \\
&= i\left(\delta^{p}_{m}\,\delta^{q}_{n} - \delta^{p}_{n}\,\delta^{q}_{m}\right)\sigma_{p,\,\mu\lambda}{}^{\cdot}\,\sigma_{q}^{\dot\lambda\rho}\,\sigma^{m}_{\rho\nu}{}^{\cdot} \\
&= 2i\left(g_{mn}\,\delta^{\rho}_{\mu} - \sigma_{n,\,\mu\lambda}{}^{\cdot}\,\sigma_{m}^{\dot\lambda\rho}\right)\sigma^{m}_{\rho\nu}{}^{\cdot} \text{ from (3)} \\
&= -6i\,\sigma_{n,\,\mu\nu}{}^{\cdot} \text{ from (2).}
\end{aligned} \tag{6}$$

Also from (2)

$$\sigma^{\dot{\lambda}\mu}_{k}\left(\sigma_{l,\mu\nu}\,{}^{\cdot}\,\sigma^{l}_{\alpha\beta}{}^{\cdot}+\sigma_{l,\mu\beta}\,{}^{\cdot}\,\sigma^{l}_{\alpha\nu}{}^{\cdot}\right)=\delta^{\dot{\lambda}}_{\nu}\,\sigma_{k,\alpha\beta}+\delta^{\dot{\lambda}}_{\beta}\,\sigma_{k,\alpha\nu}$$

$$+\frac{i}{2}\,\epsilon_{klmn}\,\sigma^{m,\dot{\lambda}\mu}\left(\sigma^{n}_{\mu\nu}{}^{\cdot}\,\sigma^{l}_{\alpha\beta}{}^{\cdot}+\sigma^{n}_{\mu\beta}{}^{\cdot}\,\sigma^{l}_{\alpha\nu}{}^{\cdot}\right) \quad (7)$$

Now notice that

$$\sigma^{m}_{\lambda\mu}{}^{\cdot}\,\sigma^{n}_{\alpha\beta}{}^{\cdot}-\sigma^{m}_{\alpha\beta}{}^{\cdot}\,\sigma^{n}_{\lambda\mu}{}^{\cdot}=\epsilon_{\lambda\alpha}\,\sigma^{m}_{\gamma\mu}{}^{\cdot}\,\sigma^{n,\gamma}_{\beta}{}^{\cdot}+\epsilon_{\mu\beta}^{\cdot\cdot}\,\sigma^{m}_{\alpha\rho}{}^{\cdot\cdot}\,\sigma^{n,\dot{\rho}}_{\lambda} \quad (8)$$

Therefore

$$\epsilon_{klmn}\,\sigma^{m\dot{\lambda}\mu}\left(\sigma^{n}_{\mu\nu}{}^{\cdot}\,\sigma^{l}_{\alpha\beta}{}^{\cdot}+\sigma^{n}_{\mu\beta}{}^{\cdot}\,\sigma^{l}_{\alpha\nu}{}^{\cdot}\right)$$

$$=\tfrac{1}{2}\,\epsilon_{klmn}\,\sigma^{m,\dot{\lambda}}_{\alpha}\left(\sigma^{n}_{\nu\mu}{}^{\cdot}\,\sigma^{l,\mu}_{\beta}{}^{\cdot\cdot}+\sigma^{n}_{\beta\mu}{}^{\cdot}\,\sigma^{l,\mu}{}^{\cdot}{}_{\dot{\nu}}\right)$$

$$=\tfrac{1}{4}\,\epsilon_{klmn}\Big[\left(\delta^{\dot{\lambda}}_{\nu}\,\sigma^{m}_{\alpha\rho}{}^{\cdot}\,\sigma^{n,\rho\dot{\mu}}\,\sigma^{l}_{\mu\beta}{}^{\cdot}-\sigma^{m}_{\nu\gamma}{}^{\cdot}\,\sigma^{n,\dot{\lambda}\gamma}\,\sigma^{l}_{\alpha\beta}{}^{\cdot}\right)$$

$$+\left(\delta^{\dot{\lambda}}_{\beta}\,\sigma^{m}_{\alpha\rho}{}^{\cdot}\,\sigma^{n,\rho\dot{\mu}}\,\sigma^{l}_{\mu\nu}{}^{\cdot}-\sigma^{m}_{\beta\gamma}{}^{\cdot}\,\sigma^{n,\dot{\lambda}\gamma}\,\sigma^{l}_{\alpha\nu}{}^{\cdot}\right)\Big]$$

$$=\tfrac{1}{2}i\left(\delta^{\dot{\lambda}}_{\nu}\,\sigma_{k,\alpha\beta}+\delta^{\dot{\lambda}}_{\beta}\,\sigma_{k\,\alpha\nu}\right)+\tfrac{1}{4}\,\epsilon_{klmn}\,\sigma^{m,\dot{\lambda}\mu}\left(\sigma^{n}_{\mu\nu}{}^{\cdot}\,\sigma^{l}_{\alpha\beta}{}^{\cdot}+\sigma^{n}_{\mu\beta}{}^{\cdot}\,\sigma^{l}_{\alpha\nu}{}^{\cdot}\right)$$

from (6)

Therefore

$$\epsilon_{klmn}\,\sigma^{m,\dot{\lambda}\mu}\left(\sigma^{n}_{\mu\nu}{}^{\cdot}\,\sigma^{l}_{\alpha\beta}{}^{\cdot}+\sigma^{n}_{\mu\beta}{}^{\cdot}\,\sigma^{l}_{\alpha\nu}{}^{\cdot}\right)=2i\left(\delta^{\dot{\lambda}}_{\gamma}\,\sigma_{k,\alpha\beta}+\delta^{\dot{\lambda}}_{\beta}\,\sigma_{k,\alpha\nu}\right) \quad (9)$$

From (7) and (9) it follows that

$$\sigma^{\dot{\lambda}\mu}_{k}\left(\sigma_{l,\mu\nu}\,{}^{\cdot}\,\sigma^{l}_{\alpha\beta}{}^{\cdot}+\sigma_{l,\mu\beta}\,{}^{\cdot}\,\sigma^{l}_{\alpha\nu}{}^{\cdot}\right)=0.$$

Multiplying by $\sigma^{k}_{\rho\dot{\lambda}}$ and using (2) one gets

$$\sigma_{l,\mu\nu}\,{}^{\cdot}\,\sigma^{l}_{\alpha\beta}{}^{\cdot}+\sigma_{l,\mu\beta}\,{}^{\cdot}\,\sigma^{l}_{\alpha\nu}{}^{\cdot}=0. \quad (10\,a)$$

Now

$$\sigma_{l,\mu\nu}\,{}^{\cdot}\,\sigma^{l}_{\alpha\beta}{}^{\cdot}-\sigma_{l,\mu\beta}\,{}^{\cdot}\,\sigma^{l}_{\alpha\nu}{}^{\cdot}=\epsilon_{\nu\beta}^{\cdot\cdot}\,\sigma_{l,\mu\rho}\,{}^{\cdot}\,\sigma^{l\dot{\rho}}_{\alpha}$$

$$=4\epsilon_{\nu\beta}^{\cdot\cdot}\,\epsilon_{\mu\alpha} \text{ from (2)} \quad (10\,b)$$

(5) follows immediately from (10).

A few other useful relations can be derived from (2). Put

$$\sigma^{k}_{\mu\lambda}{}^{\dot{}}\,\sigma^{l,\dot{\lambda}\rho}\,\sigma^{m}_{\rho\nu}{}^{\dot{}}=A^{klm}_{\mu\nu}{}^{\dot{}}$$

Then from (3)

$$
\begin{aligned}
A^{klm}_{\mu\dot{\nu}} &= A^{klm}_{\mu\dot{\nu}} \\
&= -A^{lkm}_{\mu\dot{\nu}} + 2g^{kl}\sigma^{m}_{\mu\dot{\nu}} \\
&= A^{lmk}_{\mu\dot{\nu}} + 2g^{kl}\sigma^{m}_{\mu\dot{\nu}} - 2g^{km}\sigma^{l}_{\mu\dot{\nu}} \\
&= -A^{mlk}_{\mu\dot{\nu}} + 2g^{kl}\sigma^{m}_{\mu\dot{\nu}} - 2g^{km}\sigma^{l}_{\mu\dot{\nu}} + 2g^{lm}\sigma^{k}_{\mu\dot{\nu}} \\
&= A^{mkl}_{\mu\dot{\nu}} \qquad\qquad - 2g^{km}\sigma^{l}_{\mu\dot{\nu}} + 2g^{lm}\sigma^{k}_{\mu\dot{\nu}} \\
&= -A^{kml}_{\mu\dot{\nu}} \qquad\qquad\qquad\qquad + 2g^{lm}\sigma^{k}_{\mu\dot{\nu}}
\end{aligned}
$$

Adding up the various expressions on the right one gets

$$6A^{klm}_{\mu\dot{\nu}} = -\epsilon^{klmn}\epsilon_{npqr}A^{pqr}_{\mu\dot{\nu}} + 6\left(g^{kl}\sigma^{m}_{\mu\dot{\nu}} - g^{km}\sigma^{l}_{\mu\dot{\nu}} + g^{ln}\sigma^{k}_{\mu\dot{\nu}}\right)$$

so that from (6)

$$\sigma^{k}_{\mu\dot{\lambda}}\sigma^{l,\dot{\lambda}\rho}\sigma^{m}_{\rho\dot{\nu}} = g^{kl}\sigma^{m}_{\mu\dot{\nu}} - g^{km}\sigma^{l}_{\mu\dot{\nu}} + g^{lm}\sigma^{k}_{\mu\dot{\nu}} + i\epsilon^{klmn}\sigma_{n,\mu\dot{\nu}} \tag{11 a}$$

The conjugate-complex equation is

$$\sigma^{k}_{\dot{\mu}\lambda}\sigma^{l,\lambda\dot{\rho}}\sigma^{m}_{\dot{\rho}\nu} = g^{kl}\sigma^{m}_{\dot{\mu}\nu} - g^{km}\sigma^{l}_{\dot{\mu}\nu} + g^{lm}\sigma^{k}_{\dot{\mu}\nu} - i\epsilon^{klmn}\sigma_{n,\dot{\mu}\nu} \tag{11 b}$$

From (3 *c*) and (11)

$$\sigma^{k}_{\mu\dot{\lambda}}\sigma^{l,\dot{\lambda}\rho}\sigma^{m}_{\rho\dot{\nu}}\sigma^{n,\dot{\nu}\mu} = 2\left(g^{kl}g^{mn} - g^{km}g^{ln} + g^{kn}g^{lm} + i\epsilon^{klmn}\right) \tag{12 a}$$

$$\sigma^{k}_{\dot{\mu}\lambda}\sigma^{l,\lambda\dot{\rho}}\sigma^{m}_{\dot{\rho}\nu}\sigma^{n,\nu\dot{\mu}} = 2\left(g^{kl}g^{nm} - g^{km}g^{lm} + g^{kn}g^{lm} - i\epsilon^{klmn}\right) \tag{12 b}$$

From (4) and (12) the following well-known relations for any anti-symmetric tensor F_{kl} (Laporte and Uhlenbeck, 1931; Fierz and Pauli, 1939) are obtained immediately.

$$F_{mn} = \tfrac{1}{4}\sigma^{\alpha\dot{\lambda}}{}_{m}\,\sigma^{\beta\dot{\mu}}{}_{n}\left(\epsilon_{\alpha\beta}f_{\dot{\lambda}\dot{\mu}} + \epsilon_{\dot{\lambda}\dot{\mu}}f_{\alpha\beta}\right) \tag{13 a}$$

$$f_{\mu}^{\ \rho}\,\sigma^{m,\mu\dot{\nu}}\sigma^{n}_{\dot{\nu}\rho} = 4F^{+mn} \tag{13 b}$$

$$f_{\dot{\mu}}^{\ \dot{\rho}}\,\sigma^{m\mu\dot{\nu}}\sigma^{n}_{\nu\dot{\rho}} = 4F^{-mn} \tag{13 c}$$

where

$$f_{\mu}^{\rho} = \tfrac{1}{2} F_{kl} \sigma^{k}_{\mu\lambda\cdot} \sigma^{l,\dot{\lambda}\rho} = \tfrac{1}{2} F^{+}_{kl} \sigma^{k}_{\mu\lambda\cdot} \sigma^{l,\dot{\lambda}\rho} \tag{14 a}$$

$$f_{\mu}^{\dot{\rho}} = \tfrac{1}{2} F_{kl} \sigma^{k}_{\cdot\mu\lambda} \sigma^{l,\lambda\dot{\rho}} = \tfrac{1}{2} F^{-}_{kl} \sigma^{k}_{\cdot\mu\lambda} \sigma^{l,\lambda\dot{\rho}} \tag{14 b}$$

and

$$F^{+}_{kl} = \tfrac{1}{2}(F_{kl} - i F^{\times}_{kl})$$

$$F^{-}_{kl} = \tfrac{1}{2}(F_{kl} + i F^{\times}_{kl})$$

$$F^{\times}_{kl} = \tfrac{1}{2} \epsilon_{klmn} F^{mn}$$

It is yet to be proved from (2) that for every proper Lorentz transformation there exists a spinor transformation such that the two applied together leave $\sigma^{k}_{\lambda\mu\cdot}$ unchanged. For this purpose it is sufficient to consider infinitesimal transformations. An infinitesimal Lorentz transformation is given by

$$x'_{k} = x_{k} + \epsilon^{l}_{k} x_{l} \quad (\epsilon_{kl} = -\epsilon_{lk}) \tag{15}$$

where ϵ_{kl} is a real infinitesimal quantity. Similarly an infinitesimal spinor transformation

$$a'_{\mu} = a_{\mu} + \eta^{\nu}_{\mu} a_{\nu} \quad (\eta_{\mu\nu} = \eta_{\mu\nu}) \tag{16 a}$$

$$a'_{\dot{\mu}} = a_{\dot{\mu}} + \eta^{\dot{\nu}}_{\dot{\mu}} a_{\dot{\nu}} \quad (\eta_{\dot{\mu}\dot{\nu}} = \overline{\eta_{\mu\nu}}) \tag{16 b}$$

is characterised by an infinitesimal spinor $\eta_{\mu\nu}$. The symmetry of $\eta_{\mu\nu}$ in μ, ν follows from the invariance of $\epsilon_{\mu\nu}$. It is therefore sufficient to show that for every ϵ^{kl} there exists a $\eta_{\mu\nu}$ such that

$$\epsilon^{kl} \sigma_{l,\alpha\dot{\lambda}} + \eta^{\beta}_{\alpha} \sigma^{k}_{\beta\dot{\lambda}} + \eta^{\dot{\mu}}_{\dot{\lambda}} \sigma^{k}_{\alpha\dot{\mu}} = 0. \tag{17}$$

From (5) and (17) the solution is easily obtained.

$$\eta_{\alpha\beta} = \tfrac{1}{4} \epsilon^{kl} \sigma_{k,\alpha\dot{\lambda}} \sigma^{\dot{\lambda}}_{l,\beta} \quad \eta_{\dot{\alpha}\dot{\beta}} = \overline{\eta_{\alpha\beta}} = \tfrac{1}{4} \epsilon^{kl} \sigma_{k,\dot{\alpha}\lambda} \sigma^{\lambda}_{l,\dot{\beta}} \tag{18}$$

The transformation matrix of (16) is therefore

$$\delta^{\beta}_{\alpha} + \tfrac{1}{4} \epsilon^{kl} \sigma_{k,\alpha\dot{\lambda}} \sigma^{\dot{\lambda}\beta}_{l} \tag{19a}$$

$$\delta^{\dot{\beta}}_{\dot{\alpha}} + \tfrac{1}{4} \epsilon^{kl} \sigma_{k,\dot{\alpha}\lambda} \sigma^{\lambda\dot{\beta}}_{l} \tag{19b}$$

It must now be shown that the transformation (19) considered as a representation of the Lorentz group satisfies the integrability conditions (see Van der Waerden, 1932). In any representation of the Lorentz group the transformation (15) is represented by

$$1+\tfrac{1}{2}\,\epsilon^{kl}\,\mathrm{I}_{kl}$$

where $\mathrm{I}^{kl}=-\mathrm{I}^{lk}$ are the representative matrices for the infinitesimal transformations (see Van der Waerden *l.c.*). Therefore in our case, from (19)

$$(\mathrm{I}_{kl})_{\alpha}^{\ \beta}=\tfrac{1}{4}\,(\sigma_{k,\,\alpha\dot{\lambda}}\,\sigma_{l}^{\ \dot{\lambda}\beta}-\sigma_{l,\,\alpha\dot{\lambda}}\,\sigma_{k}^{\ \dot{\lambda}\beta}) \tag{20}$$

where α and β on the left side are to be looked upon as matrix indices. The integrability conditions for the Lorentz group are well known and are

$$\mathrm{I}^{kl}\,\mathrm{I}^{mn}-\mathrm{I}^{mn}\,\mathrm{I}^{kl}=-g^{km}\,\mathrm{I}^{ln}+g^{lm}\,\mathrm{I}^{kn}+g^{kn}\,\mathrm{I}^{lm}-g^{ln}\,\mathrm{I}^{km} \tag{21}$$

Now from (4) and (20)

$$(\mathrm{I}_{kl})_{\alpha}^{\ \beta}=-\frac{i}{2}\,\epsilon_{klmn}\,(\mathrm{I}^{mn})_{\alpha}^{\ \beta}=-\frac{i}{4}\,\epsilon_{klmn}\,\sigma^{m}_{\ \alpha\dot{\lambda}}\,\sigma^{n,\dot{\lambda}\beta}$$

Therefore

$$(\mathrm{I}_{kl}\,\mathrm{I}_{mn}-\mathrm{I}_{mn}\,\mathrm{I}_{kl})_{\alpha}^{\ \gamma}=-\frac{1}{16}\,(\epsilon_{klpq}\,\epsilon_{mnrs}-\epsilon_{mnpq}\,\epsilon_{klrs})\,\sigma^{p}_{\ \alpha\dot{\lambda}}\,\sigma^{q\dot{\lambda}\beta}\,\sigma^{r}_{\ \beta\dot{\mu}}\,\sigma^{s,\dot{\mu}\gamma}$$

$$=-\frac{1}{16}\,\epsilon_{klpq}\,\epsilon_{mnrs}\,\{\sigma^{k}_{\ \alpha\dot{\lambda}}\,\sigma^{q,\dot{\lambda}\beta}\,\sigma^{r}_{\ \beta\dot{\mu}}\,\sigma^{s,\dot{\mu}\gamma}-\sigma^{s}_{\ \alpha\dot{\lambda}}\,\sigma^{r,\dot{\lambda}\beta}\,\sigma^{q}_{\ \beta\dot{\mu}}\,\sigma^{p,\dot{\mu}\gamma}\}$$

$$=-\frac{1}{16}\,\epsilon_{klpq}\,\epsilon_{mnrs}\,\{-2g^{pr}\,\sigma^{q}_{\ \alpha\dot{\mu}}\,\sigma^{s,\dot{\mu}\gamma}+i\,\epsilon^{pqrt}\,\sigma_{t,\,\alpha\dot{\mu}}\,\sigma^{s,\dot{\mu}\gamma}$$

$$+2\,\sigma^{s}_{\ \alpha\dot{\lambda}}\,g^{rp}\,\sigma^{q,\dot{\lambda}\gamma}+i\,\sigma^{s}_{\ \alpha\dot{\lambda}}\,\epsilon^{rqpt}\,\sigma_{t}^{\ \dot{\lambda}\gamma}\}\ \text{from (11)}$$

$$=-\frac{1}{16}\,\epsilon_{klpq}\,\epsilon_{mnrs}\,\{-8\,g^{pr}\,(\mathrm{I}^{qs})_{\alpha}^{\ \gamma}+4\,i\,\epsilon^{pqrt}\,(\mathrm{I}^{s}_{\ t})_{\alpha}^{\ \gamma}\}$$

Therefore

$$\mathrm{I}_{kl}\,\mathrm{I}_{mn}-\mathrm{I}_{mn}\,\mathrm{I}_{kl}=\tfrac{1}{2}\,\epsilon^{r}_{\ klq}\,\epsilon_{mnrs}\,\mathrm{I}^{qs}-\frac{i}{4}\,\epsilon_{klpq}\,\epsilon_{mnrs}\,\epsilon^{pqrt}\,\mathrm{I}^{s}_{\ t}$$

$$=-\tfrac{1}{2}\,(\sum_{klq}\pm g_{kln}\,g_{ln}\,g_{qs})\,\mathrm{I}^{qs}+\frac{i}{2}\,\epsilon_{mnrs}\,(\delta^{r}_{k}\,\delta^{t}_{l}-\delta^{r}_{l}\,\delta^{t}_{k})\,\mathrm{I}^{t}_{\ s}$$

Where the sum is to be taken over all permutations of *k, l, q* with + or − sign according as the permutation is even or odd. So

$$\mathrm{I}_{kl}\,\mathrm{I}_{mn}-\mathrm{I}_{mn}\,\mathrm{I}_{kl}=-\tfrac{1}{2}\,(\sum_{klq}\pm g_{km}\,g_{ln}\,g_{qs})\,\mathrm{I}^{qs}+\frac{i}{2}\,(\epsilon_{mnks}\,\mathrm{I}^{s}_{\ l}-\epsilon_{mnls}\,\mathrm{I}^{s}_{\ k})$$

$$=\tfrac{1}{2}\,[-g_{km}\,\mathrm{I}_{ln}+g_{lm}\,\mathrm{I}_{kn}+g_{kn}\,\mathrm{I}_{lm}-g_{lm}\,\mathrm{I}_{kn}]$$

$$+\tfrac{1}{4}\,(\epsilon_{mnks}\,\epsilon^{s}_{\ lpq}\,\mathrm{I}^{pq}-\epsilon_{mnls}\,\epsilon^{k}_{\ spq}\,\mathrm{I}^{pq})$$

$$=[-g_{km}\,\mathrm{I}_{ln}+g_{lm}\,\mathrm{I}_{kn}+g_{kn}\,\mathrm{I}_{lm}-g_{ln}\,\mathrm{I}_{km}]$$

Therefore (21) is fulfilled.

It is now possible to consider finite Lorentz transformations. Let the transformations (15) be denoted by $1+\frac{1}{2}\epsilon_{kl}J^{kl}$. The transformation matrix of (15) is

$$\delta^l_k + \epsilon^l_k$$

Therefore

$$\tfrac{1}{2}(\epsilon_{kl}J^{kl})^n_m = \epsilon^n_m \tag{22}$$

where m and n are the matrix indices of the transformation J^{kl}. A finite Lorentz transformation L can be generated from the infinitesimal transformation by the following common device.

$$\mathrm{L} = \left(1+\tfrac{1}{2}\frac{\theta^{kl}}{n}J_{kl}\right)^n_{\mathrm{Lim}\, n\to\infty} = e^{\frac{1}{2}\theta^{kl}J_{kl}}$$

$$= 1+\tfrac{1}{2}\theta^{kl}J_{kl}+\frac{(\frac{1}{2}\theta^{kl}J_{kl})^2}{2\,!}+\frac{(\frac{1}{2}\theta^{kl}J_{kl})^3}{3\,!}+\cdots$$

On account of (22) one can write symbolically

$$\mathrm{L} = e^\theta = 1+\theta+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}+\cdots$$

where θ^n is a matrix defined by induction as follows:

$$(\theta^{n+1})_p{}^q = (\theta^n)_p{}^r\,\theta_r{}^q$$

It is easy to verify that

$$\theta_{kl}\,\theta^{lm}\,\theta_{mn}\,\theta^{np}+\tfrac{1}{2}(\theta_{mn}\,\theta^{mn})\,\theta_{kl}\,\theta^{lp}-(\tfrac{1}{8}\theta^{lm}\,\theta^{nr}\,\epsilon_{lmnr})^2\,\delta^p_k = 0$$

for any tensor θ_{kl} whose only non-vanishing components are $\theta_{03} = -\theta_{30}$ and $\theta_{12} = -\theta_{21}$. However since every antisymmetrical tensor can be brought to this form by a Lorentz transformation it follows that the above identity is valid for every θ_{kl}. Written in matrix form it becomes

$$\theta^4+\tfrac{1}{2}\theta_{kl}\,\theta^{kl}\,\theta^2-(\tfrac{1}{8}\theta^{kl}\,\theta^{mn}\,\epsilon_{klmn})^2 = 0$$

Making use of this characteristic equation of θ, it can be proved that

$$\mathrm{L} = \tfrac{1}{2}[\cosh\sqrt{\Phi_+}+\cosh\sqrt{\Phi_-}]+\frac{\cosh\sqrt{\Phi_+}-\cosh\sqrt{\Phi_-}}{\Phi_+-\Phi_-}(\theta^2+\tfrac{1}{4}\theta_{kl}\,\theta^{kl})$$

$$+\tfrac{1}{2}\left[\frac{\sinh\sqrt{\Phi_+}}{\sqrt{\Phi_+}}+\frac{\sinh\sqrt{\Phi_-}}{\sqrt{\Phi_-}}\right]\theta$$

$$+\left(\frac{\sinh\sqrt{\Phi_+}}{\sqrt{\Phi_+}}-\frac{\sinh\sqrt{\Phi_-}}{\sqrt{\Phi_-}}\right)\frac{1}{\Phi_+-\Phi_-}(\theta^3+\tfrac{1}{4}\theta_{kl}\,\theta^{kl}\cdot\theta) \tag{23 a}$$

where

$$\Phi_\pm = -\tfrac{1}{4}\theta_{kl}\,\theta^{kl}\pm\tfrac{1}{4}\sqrt{(\tfrac{1}{2}\theta^{kl}\,\theta^{mn}\,\epsilon_{klmn})^2+(\theta^{kl}\,\theta_{kl})^2} \tag{23 b}$$

This is the expression for the most general proper Lorentz transformation.

The spinor transformation Λ associated to (23) is given by

$$\Lambda = \left(1+\tfrac{1}{2}\frac{\theta^{kl}\mathrm{I}_{kl}}{n}\right)^n_{\mathrm{Lim}\, n\to\infty} = e^{\frac{1}{2}\theta^{kl}\mathrm{I}_{kl}}$$

$$= 1+(\tfrac{1}{2}\theta^{kl}\mathrm{I}_{kl})+\frac{(\frac{1}{2}\theta^{kl}\mathrm{I}_{kl})^2}{2\,!}+\cdots\cdots \tag{24}$$

Now

$$[(\tfrac{1}{2}\theta^{kl}\,\mathrm{I}_{kl})^2]^{\gamma}_{\alpha} = (\tfrac{1}{2}\theta^{kl}\,\mathrm{I}_{kl})^{\beta}_{\alpha}(\tfrac{1}{2}\theta^{mn}\,\mathrm{I}_{mn})^{\gamma}_{\beta}$$

$$= \frac{1}{16}\theta^{kl}\,\theta^{mn}\,\sigma_{k,\alpha\dot\lambda}\,\sigma_{l}^{\dot\lambda\beta}\,\sigma_{m,\beta\dot\mu}\,\sigma_{n}^{\dot\mu\gamma}$$

$$= \frac{1}{16}\,\theta_{kl}\,\theta_{mn}\,(-2g^{km}\,\sigma^{l}_{\alpha\dot\mu}\,\sigma^{n,\dot\mu\gamma} + i\,\epsilon^{klmp}\,\sigma_{p,\alpha\dot\mu}\,\sigma^{n,\dot\mu\gamma})\text{ from (11)}$$

$$= -\tfrac{1}{8}\,\theta_{kl}\,\theta^{kl}\,\delta^{\gamma}_{\alpha} + \frac{i}{16}\,\theta_{kl}\,\theta_{mn}\,\epsilon^{klmp}\,\{\delta^{n}_{p}\,\delta^{\gamma}_{\alpha} - i\,\epsilon^{n}_{prs}\,(\mathrm{I}^{rs})^{\gamma}_{\alpha}\}\text{ from (2)}$$

$$= (-\tfrac{1}{8}\,\theta_{kl}\,\theta^{kl} + \frac{i}{16}\,\theta_{kl}\,\theta_{mn}\,\epsilon^{klmn})\,\delta^{\gamma}_{\alpha}$$

Put

$$\Theta = -\tfrac{1}{8}\,\theta_{kl}\,\theta^{kl} + \frac{i}{16}\,\theta_{kl}\,\theta_{mn}\,\epsilon^{klmn} \tag{25}$$

Then from (24),

$$\Lambda = \left(1+\frac{\Theta}{2\,!}+\frac{\Theta^2}{4\,!}+\cdots\right)+\left(1+\frac{\Theta}{3\,!}+\frac{\Theta^2}{5\,!}+\cdots\right)(\tfrac{1}{2}\,\theta^{kl}\,\mathrm{I}_{kl})$$

$$= \cosh\sqrt{\Theta} + \frac{\sinh\sqrt{\Theta}}{\sqrt{\Theta}}\,(\tfrac{1}{2}\,\theta^{kl}\,\mathrm{I}_{kl}) \tag{26 a}$$

or

$$\Lambda^{\beta}_{\alpha} = \cosh\sqrt{\Theta}\;\delta^{\beta}_{\alpha} + \frac{\sinh\sqrt{\Theta}}{\sqrt{\Theta}}\,\tfrac{1}{4}\,\theta^{kl}\,\sigma_{k,\alpha\dot\lambda}\,\sigma_{l}^{\dot\lambda\beta} \tag{26b}$$

Therefore the spinor transformation associated to any given Lorentz transformation is completely determined. As an example consider the case of a spatial rotation about the x_3 axis. In this all components of θ_{kl} are zero except $\theta_{12} = -\theta_{21} = 2\phi$ (say). Therefore

$$\Theta = -\phi^2$$

and

$$\Lambda^{\alpha}_{\beta} = \cos\phi\;\delta^{\beta}_{\alpha} + i\,\frac{\sin\phi}{i\phi}\,\frac{\phi}{2}\,(\sigma_{1,\alpha\dot\lambda}\,\sigma_{2}^{\dot\lambda\beta} - \sigma_{2,\alpha\dot\lambda}\,\sigma_{1}^{\dot\lambda\beta})$$

$$= \cos\phi\;\delta^{\beta}_{\alpha} + \frac{\sin\phi}{i}\,\tfrac{1}{2}\,\epsilon_{1203}\,(\sigma^{0}_{\alpha\dot\lambda}\,\sigma^{3\,\dot\lambda\beta} - \sigma^{3}_{\alpha\dot\lambda}\,\sigma^{0,\dot\lambda\beta})$$

$$= \cos\phi\;\delta^{\beta}_{\alpha} + i\sin\phi\,\tfrac{1}{2}\,(\sigma^{0}_{\alpha\dot\lambda}\,\sigma^{3,\dot\lambda\beta} - \sigma^{3}_{\alpha\dot\lambda}\,\sigma^{0,\dot\lambda\beta}) \tag{27 a}$$

Similarly for a Lorentz transformation along the x_3 axis the only non-vanishing component of θ_{kl} is $\theta_{03} = -\theta_{30} = 2\phi$ (say), so that $\Theta = \phi^2$ and

$$\Lambda^{\beta}_{\alpha} = \mathrm{cash}\;\phi\cdot\delta^{\beta}_{\alpha} - \sinh\phi\cdot\tfrac{1}{2}\,(\sigma_{0,\alpha\dot\lambda}\,\sigma_{3}^{\dot\lambda\beta} - \sigma_{3,\alpha\dot\lambda}\,\sigma_{0}^{\dot\lambda\beta}) \tag{27 b}$$

In the usual representation σ^0 is the unit matrix and $\sigma^{3,\dot\alpha\beta} = -\sigma^3_{\dot\alpha\beta}$. Therefore in matrix notation (27*a*) and (27*b*) can be written as

$$\Lambda = \cos\phi - i\sin\phi\,\sigma^3$$
$$\Lambda = \cosh\phi - \sinh\phi\sigma^3$$

where the matrix elements of σ^3 are $\sigma^3_{\dot\alpha\beta}$.

Till now only the proper Lorentz transformations have been discussed. Reflection can now be included in the following way. From (11)

$$\sigma^{0,\dot\lambda}_{\mu}\,\sigma^{k}_{\dot\lambda\rho}\,\sigma^{0,\rho}_{\dot\nu} = 2g^{0k}\,\sigma^{0}_{\mu\dot\nu} - \sigma^{k}_{\mu\dot\nu} = g^{kk}\,\sigma^{k}_{\mu\dot\nu} \tag{28}$$

Therefore $\sigma^{0,\dot\lambda}_{\mu}$ is the reflection matrix, and by reflection the spinors $a_{\dot\mu}$ and b_μ go over into a_μ and $b_{\dot\mu}$ given by

$$a_\mu = \sigma^{0,\dot\lambda}_{\mu}\,a_{\dot\lambda}, \quad b_{\dot\mu} = \sigma^{0,\lambda}_{\dot\mu}\,b_\lambda \tag{29}$$

The quantities $\sigma^k_{\mu\dot\nu}$ remain unchanged for the simultaneous application of reflection to the tensor index k as well as to the spinor indices μ, ν. In the usual representation $\sigma^0_{\mu\dot\nu}$ is the unit matrix and therefore the only non-vanishing components of $\sigma^{0,\dot\lambda}_{\mu}$ are $\sigma^{0,\dot2}_{1} = -\sigma^{0,\dot1}_{2} = -1$. In this case therefore (29) coincides with the usual rules for reflection.

If $\sigma^k_{\lambda\dot\mu}$ and $\sigma'^k_{\lambda\dot\mu}$ be two different sets of σ's satisfying (1) and (2) then it follows from (5) that

$$\sigma'^{k}_{\lambda\dot\mu} = a^k_l\,\sigma^l_{\lambda\dot\mu}$$

where

$$a^k_l = \tfrac{1}{2}\sigma'^{k}_{\lambda\dot\mu}\,\sigma^{\dot\lambda\mu}_{l}$$

so that a^k_l are real and

$$a^k_m\,a^l_n\,g^{mn} = g^{kl}$$

Therefore a^k_l must be the coefficients of a Lorentz transformation apart from the fact that they may reverse the direction of time.

Since the equations (1) and (2) are in a proper covariant form they remain valid for all real transformations of the tensor space and any arbitrary transformations of the spin-space (*cf.* Infeld and Van der Waerden, 1933). However it must be borne in mind that for this general case

$$\epsilon_{0123} = -\sqrt{-g}$$
$$\epsilon^{0123} = \frac{1}{\sqrt{-g}}$$

where g is the determinant of the general g_{ik} matrix. Also ϵ_{12} and ϵ^{12} are no longer 1 but are equal to γ and $\frac{1}{\gamma}$ respectively where γ is a spinor density of weight 1, *i.e.*, on transformation it gets multiplied by the determinant of the transformation in the spin-space (Infeld and Van der Waerden, 1933). All the results [*e.g.*, equations (5), (11) and (12)] therefore hold also for the general case which is of importantce in the general theory of relativity.

In conclusion let us consider an interesting application of (11) to the Dirac equation of a particle of spin $\frac{1}{2}$. Expressed in terms of spinors it splits up into the following two equations:

$$i\,\partial^{\alpha\dot{\lambda}}\,a_{\dot{\lambda}} = \chi b^{\alpha} \tag{30 a}$$

$$i\,\partial_{\alpha\dot{\lambda}}\,b^{\alpha} = \chi a_{\dot{\lambda}} \tag{30 b}$$

To these are to be added the corresponding conjugate-complex equations. Apart from numerical factors the charge-current-density spinor is given by

$$S_{\alpha\dot{\beta}} = a_{\alpha}\,a_{\dot{\beta}} + b_{\alpha}\,b_{\dot{\beta}} \tag{31}$$

where a_{α} and $b_{\dot{\beta}}$ are the complex-conjugates of $a_{\dot{\alpha}}$ and b_{β} respectively. It is obvious that the charge-density given by (21) is positive definite if the usual representation of σ's is used since in this case σ^0 is the unit matrix. For every other representation of σ's the charge-density is therefore either positive or negative definite according as this representation is obtained from the usual one by a Lorentz transformation without or with the reversal of the direction of time. However only the definite character of the charge-density is of importance, the sign being immaterial.

Notice that the equations (30) are completely equivalent to the second order equation

$$\partial_k\,\partial^k\,a_{\dot{\lambda}} + \chi^2\,a_{\dot{\lambda}} = 0 \tag{32}$$

which follows from them. This becomes obvious if one looks upon (30*a*) as the *definition* of b^{α}. Therefore (30) can be replaced by

$$\partial^k\,a_{\dot{\lambda}} = \chi\,a^{k}_{\dot{\lambda}} \tag{33 a}$$

$$\partial^k\,a^{k}_{\dot{\lambda}} = -\chi\,a^{\dot{\lambda}} \tag{33 b}$$

so that

$$b^{\alpha} = i\,\sigma_k^{\alpha\dot{\lambda}}\;a^{k}_{\dot{\lambda}}$$

(31) now becomes

$$S_{\alpha\dot{\beta}} = a_{\alpha}\,a_{\dot{\beta}} + \sigma^{l}_{\alpha\dot{\lambda}}\;\sigma^{m}_{\dot{\beta}\mu}\;a^{\dot{\lambda}}_{l}\;a^{\mu}_{m},\;(a^{\mu}_{m} = \overline{a^{\dot{\mu}}_{m}})$$

or

$$S^{k} \equiv \tfrac{1}{2}\,\sigma^{k\alpha\dot{\beta}}\,S_{\alpha\dot{\beta}} = \tfrac{1}{2}\,\sigma^{k,\,\alpha\dot{\beta}}\,a_{\alpha}\,a_{\dot{\beta}} + \tfrac{1}{2}\,\sigma^{l}_{\dot{\lambda}\alpha}\,\sigma^{k,\,\alpha\dot{\beta}}\,\sigma^{m}_{\dot{\beta}\mu}\,a^{\dot{\lambda}}_{l}\,a^{\mu}_{m}$$

$$= \tfrac{1}{2}\,\sigma^{k,\,\alpha\dot{\beta}}\,a_{\alpha}\,a_{\dot{\beta}} + \tfrac{1}{2}\,[\sigma^{m}_{\dot{\lambda}\mu}\,(a^{\mu}_{m}\,a^{k\dot{\lambda}} + a^{\dot{\lambda}}_{m}\,a^{k\mu}) - \sigma^{k}_{\dot{\lambda}\mu}\,a^{m\dot{\lambda}}\,a^{\mu}_{m}]$$

$$+\, i\,\epsilon^{klmn}\,a^{\dot{\lambda}}_{l}\,a^{\mu}_{m}\,\sigma_{n,\,\dot{\lambda}\mu} \quad \text{from (11)} \tag{34}$$

The equations (33) and (34) are completely equivalent to the usual formulation of the Dirac-equation in the force-free case. The definite character of the charge-density is not quite obvious from (34). Equations (33) resemble very much the corresponding equations for a particle of spin 0.

However the equivalence of (30) and (33) holds only for the force-free case. In case of interaction with an electromagnetic field (30) go over into

$$i\,\pi^{\alpha\dot{\lambda}}\,a_{\dot{\lambda}} = \chi\,b^{\alpha} \tag{35 a}$$

$$i\,\pi_{\alpha\dot{\lambda}}\,b^{\alpha} = \chi\,a_{\dot{\lambda}} \tag{35 b}$$

where $\pi^{\alpha\dot{\lambda}}$ is the spinor corresponding to $\pi_k \equiv \partial_k + ie\,\phi_k$, ϕ_k being the electromagnetic potentials and e the charge of the particle. The second order equation derived from (35) is

$$\pi_{\alpha\dot{\lambda}}\,\pi^{\alpha\dot{\mu}}\,a_{\dot{\mu}} + \chi^2\,a_{\dot{\lambda}} = 0$$

Or from (3)

$$\pi_{k}\,\pi^{k}\,a_{\dot{\lambda}} + \tfrac{1}{2}\,(\pi_{k}\,\pi_{l} - \pi_{l}\,\pi_{k})\,\sigma^{k}_{\alpha\dot{\lambda}}\,\sigma^{l,\,\alpha\dot{\mu}}\,a_{\dot{\mu}} + \chi^2\,a_{\dot{\lambda}} = 0$$

which is not the same as that obtained by replacing ∂_k by π_k in (32). Therefore to take electromagnetic interaction into account it is not sufficient to replace ∂^k by π^k in (33). The correct generalisation of (33) in this case is

$$\pi^{k}\,a_{\dot{\lambda}} = \chi\,a^{k}_{\dot{\lambda}} \tag{36 a}$$

$$\pi_{k}\,a^{k}_{\dot{\lambda}} + \frac{ie}{\chi}\,f^{\dot{\mu}}_{\dot{\lambda}}\,a_{\dot{\mu}} = -\,\chi a_{\dot{\lambda}} \tag{36 b}$$

where

$$f^{\dot{\mu}}_{\dot{\lambda}} = \tfrac{1}{2}\,f_{kl}\,\sigma^{k}_{\dot{\lambda}\alpha}\,\sigma^{l\alpha\dot{\mu}} \quad \text{and} \quad f_{kl} = \partial_{k}\,\phi_{l} - \partial_{l}\,\phi_{k}$$

The extra term in (36 b) corresponds precisely to the magnetic moment $\frac{e}{\chi}$ of the electron in Dirac's theory. The expressions (31) and (34) for the current vector remain unchanged. From (36) it follows that the Dirac equation is ***completely equivalent*** to the second order equation

$$\pi_{k}\,\pi^{k}\,a_{\dot{\lambda}} + i\,e\,f^{\dot{\mu}}_{\dot{\lambda}}\,a_{\dot{\mu}} + \chi^2\,a_{\dot{\lambda}} = 0$$

provided the current vector is defined by (34). Equations (36) emphasise the fact that even in the simple case of spin $\frac{1}{2}$ correct electromagnetic interaction cannot be introduced simply by replacing ∂_k by π_k in any arbitrary formulation which is valid for the force-free case.

Summary

The σ-symbols are defined by means of the equations (1) and (2). All their properties are deduced from their definition without making use of any explicit representation. Certain interesting relations concerning the product of three or more σ's are obtained. They are shown to be useful in transforming tensors into spinors and *vice versa*.

Directly from (1) and (2) it is deduced that corresponding to every proper Lorentz transformation there exists a spinor-transformation such that the two applied together leave the σ's unchanged. The spinor transformation corresponding to the most general proper Lorentz transformation is explicitly given. Also the spinor-transformation corresponding to reflection is obtained. It is pointed out that since the defining equations (1) and (2) and the relations deduced from them are already in a proper covariant form they can be taken over as such to the general theory of relativity.

Finally the Dirac equation for a particle of spin $\frac{1}{2}$ is discussed from a new angle. Here only one spinor together with its space-time derivatives (and *not* two spinors) is used to describe the particle. It is shown that the Dirac equation is completely equivalent to a second order equation for this single spinor. The expression for the charge-current density in terms of this single spinor and its derivatives is obtained. In the present formulation correct electromagnetic interaction can be introduced only by the addition of an extra term depending explicitly on the field. This additional term is the one which corresponds to the magnetic moment of the electron.

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