CHARACTERS OF THE CLASSES OF THE FORM 
(n_1, n_2, n_3) IN SYMMETRIC GROUPS

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1. INTRODUCTION

In a previous paper* the author proved that the characters of the classes 
(n) and (n_1, n_2) are zero except in the following cases:

(1.1) \( \chi_{(n)}^{[x, 1^a]} = (-1)^a \) \( x \geq 1 \)

(2.1) \( \chi_{(n_1, n_2)}^{[x, 1^a]} = (-1)^a \) when \( 0 \leq a < n_2 \)
\( = (-1)^{a-1} \) when \( a \geq n_1 \)
\( = (-1)^{n_1 - 1} \) when \( a = n_2 \) or \( n_2 = 1 \) only if \( n_1 = n_2 \)

(2.2) \( \chi_{(n_1, n_2)}^{[x, y, 2^b, 1^c]} = (-1)^a \) if \( n_1 \neq n_2 \) \( \{ \text{when } x = n_1 - a - b - 1 \}
\( = 2(-1)^{b+1} \) if \( n_1 = n_2 \) \( \text{and } y = n_2 - b + 1 \)

(2.3) \( \chi_{(n_1, n_2)}^{[x, y, 2^b, 1^c]} = (-1)^a \) if \( n_1 = n_2 \) \( \{ \text{when } x = n_1 - b \}
\( = 2(-1)^a \) if \( n_1 = n_2 \) \( \text{and } y = n_2 - b - a \)

(2.4) \( \chi_{(n_1, n_2)}^{[x, y, 2^b, 1^c]} = (-1)^a \) if \( n_1 = n_2 \) \( \{ \text{when } x = n_2 - b \}
\( = 2(-1)^a \) if \( n_1 = n_2 \) \( \text{and } y = n_1 - a - b \).

For \( n_1 = n_2 \) results (2.3) and (2.4) are identical. In quoting the
above results the notation is slightly altered for the sake of uniformity.

In the present paper we obtain the characters of the classes of elements
containing three cycles.

2. CHARACTERS OF THE CLASSES OF THE FORM (n_1, n_2, n_3)

If \( P_1, P_2, P_3 \ldots \) stand respectively for the totality of the partitions

(1) \([x, 1^a]\) \( x \geq 1; a \geq 0 \)

(2) \([x, y, 2^b, 1^c]\) \( x \geq y \geq 2; a \geq 0, b \geq 0 \)

(3) \([x, y, z, 3^a, 2^b, 1^c]\) \( x \geq y \geq z \geq 3; a \geq 0, b \geq 0, c \geq 0 \)

we have the following:

**Theorem 1.**

\[ \chi^{[\lambda]} \neq 0 \]

for all partitions \([\lambda]\) not belonging to \(P_1, P_2, \ldots, P_k\).

The proof of the theorem follows immediately if we note that it is impossible to build regular graphs of the partitions not belonging to \(P_1, P_2, \ldots, P_k\) by consecutive regular applications of \(n_1, n_2, \ldots, n_k\) nodes.

From the above theorem it follows that the non-zero characters of the classes \((n_1, n_2, n_3)\) come only from partitions belonging to \(P_1, P_2, P_3\). But this does not mean that the characters are different from zero in all the partitions of \(P_1, P_2\) and \(P_3\). Using the recurrence relations between \(S\)-functions\(\dagger\) we now obtain the characters of the classes \((n_1, n_2, n_3)\) in terms of the characters of the classes \((n_1, n_2)\).

**Theorem 2.**

\[ \chi^{[x, 1^y]}_{(n_1, n_2, n_3)} = \alpha_1 + \beta_1 \]

where

\[ \alpha_1 = (-1)^{y_1-1} \chi^{[x, 1^{n_3-a}]}_{(n_1, n_2)} \]

if \(a \geq n_3\)

and

\[ \beta_1 = \chi^{[x-n_3, 1^y]}_{(n_1, n_2)} \]

if \(x > n_3\)

\(\alpha_1\) and \(\beta_1\) are zero in all other cases.

Using (2-1) we obtain \(\chi^{[x, 1^y]}_{(n_1, n_2, n_3)}\).

**Theorem 3.**

\[ \chi^{[x, y, 2^t]}_{(n_1, n_2, n_3)} = \alpha_2 + \beta_2 + \gamma_4 + \delta_2 \]

where

\[ \alpha_2 = (-1)^{n_1-1} \chi^{[x, y, 1^{n_2-b}]}_{(n_1, n_2)} \]

if \(a \geq n_3\)

\[ \beta_2 = (-1)^{y_2-1} \chi^{[x, y, 2^{n_3-a}]}_{(n_1, n_2)} \]

if \(b \geq n_3\)

\[ \beta_2' = (-1)^{x+b-t-1} \chi^{[t, y, 2^{n_3-a}]}_{(n_1, n_2)} \]

for \(t < b\) and

\[ n_3 + t = a + b + 1 \]


A3b
\[\gamma_2 = \chi_{(n_1, n_2, n_3)}^{[x, y - n \lambda, 2x, 1\varepsilon]} \quad \text{if } y \geq n_3 \pm 2\]
\[= (-1)^{\gamma} \chi_{(n_1, n_2)}^{[x, y - n \lambda + h \cdot + a]} \quad \text{if } 0 \leq y - n_3 = 1 - b\]
\[= (-1)^{\beta_2} \chi_{(n_1, n_2, n_3)}^{[x, y - n \lambda + h \cdot + a]} \quad \text{if } y - n_3 = -(a + b)\]
\[\delta_2 = \chi_{(n_1, n_2)}^{[x - n \lambda, y, 2b, 1\varepsilon]} \quad \text{if } x - n_3 \geq y\]
\[= (-1) \chi_{(n_1, n_2, n_3)}^{[y - 1, x - n \lambda - 1, 2x, 1\varepsilon]} \quad \text{if } y - 1 \geq x - n_3 \geq 1\]
\[= (-1)^{\delta_2} \chi_{(n_1, n_2, n_3)}^{[y - 1, 1 \varepsilon + h \cdot + a]} \quad \text{if } y - 1 > x - n_3 = -b.\]

\(a_2, \beta_2, \beta_2', \gamma_2, \delta_2\) are zero in all other cases.

Using (2.1) to (2.4) \(\chi_{(n_1, n_2, n_3)}^{[x, y, z, 2^\varepsilon, 1\varepsilon]}\) is obtained.

**Theorem 4.**

\[\chi_{(n_1, n_2, n_3)}^{[x, y, z, 2^\varepsilon, 1\varepsilon]} = \alpha_3 + \beta_3 + \gamma_3\]
where
\[\alpha_3 = (-1) \chi_{(n_1, n_2, n_3)}^{[x, y, z, 2^\varepsilon + 1, 1\varepsilon]} \quad \text{if } z - n_3 = 2 - c\]
\[= (-1)^{\beta_3} \chi_{(n_1, n_2, n_3)}^{[x, y, z, 2^\varepsilon + 1, 1\varepsilon]} \quad \text{if } z - n_3 = 1 - b - c\]
\[= (-1)^{\gamma_3} \chi_{(n_1, n_2, n_3)}^{[x, y, z, 2^\varepsilon + 1, 1\varepsilon]} \quad \text{if } z - n_3 = -(a + b + c)\]
\[\beta_3 = (-1)^{\beta_3} \chi_{(n_1, n_2, n_3)}^{[x, y, z - 1, 2^\varepsilon + 1, 1\varepsilon]} \quad \text{if } z - 1 \geq y - n_3 = 1 - c\]
\[= (-1)^{\beta_3} \chi_{(n_1, n_2, n_3)}^{[x, y, z - 1, 2^\varepsilon + 1, 1\varepsilon]} \quad \text{if } z - 1 > y - n_3 = -b + c\]
\[= (-1)^{\gamma_3} \chi_{(n_1, n_2, n_3)}^{[x, y, z - 1, 2^\varepsilon + 1, 1\varepsilon]} \quad \text{if } z - 1 > y - n_3 = 1 - c\]
\[\gamma_3 = (-1)^{\gamma_3} \chi_{(n_1, n_2, n_3)}^{[x, y, z, 2^\varepsilon + 1, 1\varepsilon]} \quad \text{if } z - 2 \geq x - n_3 = -c\]
\[= (-1)^{\gamma_3} \chi_{(n_1, n_2, n_3)}^{[x, y, z, 2^\varepsilon + 1, 1\varepsilon]} \quad \text{if } z - 2 > x - n_3 = -1 - b + c\]

\(a_3, \beta_3, \gamma_3\) are zero in all other cases.
Using relations (2·2) to (2·4) \( \chi_{(n_1, n_2, n_3)} \) is evaluated.

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**SUMMARY**

It is shown in this paper that the non-zero characters of the classes of the form \( (n_1, n_2, \ldots, n_k) \) come only from partitions belonging to \( P_1, P_2, \ldots, P_k \). Using the recurrence relations between S-functions the characters of the classes \( (n_1, n_2, n_3) \) are expressed in terms of the known characters of the classes \( (n_1, n_2) \).