THE LOWER ORDER OF THE ZEROS OF AN INTEGRAL FUNCTION (II)

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1. Let \( f(z) \) be an integral function of finite order \( \rho \). In a previous note\(^1\) I proved that if \( 0 < \rho < 1 \) then
\[
\lambda \leq 1 + \frac{\lambda_1}{\lambda_1 - \rho}.
\]
I prove here

**Theorem.** If \( f(z) \) be an integral function of order \( \rho \) where \( \rho > 0 \) is non-integer\(^2\) then
\[
\lambda \leq \rho + \frac{\lambda_1}{\lambda_1 \rho + \frac{\rho(\rho - 1)}{\rho + 1 - \rho}} \quad (1)
\]
where \( \rho, \lambda, \lambda_1 \) denote the genus, the lower order and the lower order of the zeros of \( f(z) \).

§2. Proof. We have\(^3\)
\[
\log M(r) < A \{ r^\rho + 1 \}
\]
where
\[
I = \int_0^\infty \frac{n(x)}{x^{\rho+1}} \frac{r^\rho + 1}{x + r} \, dx.
\]
We may suppose \( \rho > 1 \). Let\(^4\) \( \lambda_1 < \mu < \rho < \nu \). There exists an infinity of \( R \) such that \( n(R) < R^\mu \).

Let \( N = R^\mu \rho \) and \( r = N^\beta \) where
\[
\beta = \rho - \rho + \frac{\mu}{\mu} (\rho + 1 - \rho).
\]

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\(^1\) S. M. Shah, "The Lower Order of the Zeros of an Integral Function," *Journal Indian Math. Soc.*, 1942, 6, No. 2. \( \lambda, \lambda_1 \) have the same meaning.

\(^2\) When \( \rho \) is integer, the result (1) holds but is trivial.

\(^3\) A B K K \( K_1 \) \( K_2 \) are constants.

\(^4\) If \( \lambda_1 = \rho \) then the right-hand side expression of (1) becomes \( \rho \) and (1) is obviously true.
The Lower Order of the Zeros of an Integral Function (II) 163

Hence \( 1 < \beta < \frac{p}{\mu} \) and

\[
I = \int_{0}^{N} \int_{N}^{R} \int_{R}^{\infty} = I_1 + I_2 + I_3 \text{ say.}
\]

\[
I_1 < \int_{0}^{N} \frac{K x^\nu r^\theta + 1}{x^\delta + 1 (x + r)} \, dx < K_1 \, r^\theta \, N^\nu - r
\]

\[
= K_1 \exp \left\{ \left( p + \frac{\nu - p}{\beta} \right) \log r \right\}
\] (2)

\[
I_2 < \int_{N}^{R} \frac{n (R) r^\theta + 1}{x^\delta + 1 (x - r)} \, dx
\]

\[
\leq R^\mu \, r^\theta + 1 \left[ \int_{N}^{N^\beta} \frac{dx}{r x^\delta + 1} + \int_{N^\beta}^{R} \frac{dx}{x^\delta + \frac{\beta}{p}} \right]
\]

\[
< R^\mu \, r^\theta + 1 \left[ \frac{1}{pr N^\beta} + \frac{1}{p + 1} \frac{1}{N^\beta (\rho + 1)} \right]
\]

\[
< \frac{1}{p} \left[ \exp \left( p + \frac{\rho - p}{\beta} \right) \log r + \exp \left( \frac{\rho}{\beta} \log r \right) \right]
\] (3)

\[
I_3 < \int_{R}^{\infty} \frac{K x^\nu r^\theta + 1}{x^\delta + 1 (x + r)} \, dx
\]

\[
< \int_{R}^{\infty} K x^{\nu - \theta - 2} r^\theta + 1 \, dx = K_2 \, r^\theta + 1 \, R^\nu - r - 1
\]

\[
= K_2 \exp \left\{ \left( p + 1 + \frac{\rho (v - p - 1)}{\mu \beta} \right) \log r \right\}
\] (4)

Since \( \frac{\rho}{\beta} < p + \frac{\rho - p}{\beta} < p + \frac{\nu - p}{\beta} < p + 1 + \frac{\rho (v - p - 1)}{\mu \beta} \)

we obtain from (2), (3) and (4) that for an infinity of \( r \)

\[
\log M (r) < B \exp \left[ \left\{ p + 1 + \frac{\rho (v - p - 1)}{\mu \beta} \right\} \log r \right]
\]
Since $\mu - \lambda_1$ and $\nu - \rho$ may be chosen arbitrarily small we get

$$\lambda \leq p + 1 + \frac{\rho (p - p - 1)}{\lambda_1 (\rho - p) + \rho (p + 1 - \rho)}$$

$$= p + \frac{(p - \rho) \lambda_1}{\lambda_1 (\rho - p) + \rho (p + 1 - \rho)}$$

and the theorem is proved.

§3. Corollary. Given $\rho = \lambda$ and $\rho > 0$ non-integer then $\rho = \rho_1 = \lambda = \lambda_1$.

For

$$\rho = \lambda \leq p + \frac{(p - \rho) \lambda_1}{\lambda_1 (\rho - p) + \rho (p + 1 - \rho)}$$

$$\therefore (p - \rho) \leq \frac{(p - \rho) \lambda_1}{\lambda_1 (\rho - p) + \rho (p + 1 - \rho)}$$

$$\therefore (p + 1 - \rho) (\rho - \lambda_1) \leq 0$$

$$\therefore \rho \leq \lambda_1$$

Hence $\rho = \lambda_1$. 