THE LOWER ORDER OF THE ZEROS OF AN INTEGRAL FUNCTION (II)

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1. LET $f(z)$ be an integral function of finite order ρ. In a previous note[1] I proved that if $0 < \rho < 1$ then

$$\lambda \leqslant \frac{\lambda_1}{1 + \lambda_1 - \rho}$$

I prove here

THEOREM. *If* f (z) *be an integral function of order* ρ *where* $\rho > 0$ *is non-integer*[2] *then*

$$\lambda \leqslant p + \frac{(\rho - p)\,\lambda_1}{\lambda_1 (\rho - p) + \rho\,(p + 1 - \rho)} \tag{1}$$

where p, λ, λ_1 *denote the gonus, the lower order and the lower order of the zeros of* f (z).

§2. PROOF. We have[3]

$$\log M(r) < A\,\{r^p + I\} \text{ where}$$

$$I = \int_0^\infty \frac{n(x)}{x^{p+1}} \frac{r^{p+1}}{x + r}\, dx \cdot$$

We may suppose $p \geqslant 1$. Let[4] $\lambda_1 < \mu < \rho < \nu$. There exists an infinity of R such that $n(R) < R^\mu$.

Let $N = R^{\mu/\rho}$ and $r = N^\beta$ where

$$\beta = \rho - p + \frac{\rho}{\mu}(p + 1 - \rho).$$

[1] S. M. Shah, "The Lower Order of the Zeros of an Integral Function," *Journal Indian Math. Soc.*, 1942, **6**, No. 2. λ, λ_1 have the same meaning.

[2] When ρ is integer, the result (1) holds but is trivial.

[3] A B K K_1 K_2 are constants.

[4] If $\lambda_1 = \rho$ then the right-hand side expression of (1) becomes ρ and (1) is obviously true.

Hence $1 < \beta < \frac{\rho}{\mu}$ and

$$I = \int_0^N + \int_N^R + \int_R^\infty = I_1 + I_2 + I_3 \text{ say.}$$

$$I_1 < \int_0^N \frac{Kx^\nu r^{p+1}}{x^{p+1}(x+r)}\, dx < K_1 r^p N^{\nu-p}$$

$$= K_1 \exp\left\{\left(p + \frac{\nu-p}{\beta}\right) \log r\right\} \tag{2}$$

$$I_2 < \int_N^R \frac{n(R)\, r^{p+1}}{x^{p+1}(x-r)}\, dx$$

$$\leqslant R^\mu r^{p+1}\left[\int_N^{N^\beta} \frac{dx}{r\, x^{p+1}} + \int_{N^\beta}^R \frac{dx}{x^{p+2}}\right]$$

$$< R^\mu r^{p+1}\left[\frac{1}{pr}\frac{1}{N^p} + \frac{1}{p+1}\frac{1}{N^{\beta(p+1)}}\right]$$

$$< \frac{1}{p}\left[\exp\left(p + \frac{\rho-p}{\beta}\right)\log r + \exp\left(\frac{\rho}{\beta}\log r\right)\right] \tag{3}$$

$$I_3 < \int_R^\infty \frac{K\, x^\nu r^{p+1}}{x^{p+1}(x+r)}\, dx$$

$$< \int_R^\infty K\, x^{\nu-p-2}\, r^{p+1}\, dx = K_2\, r^{p+1} R^{\nu-p-1}$$

$$= K_2 \exp\left\{\left(p + 1 + \frac{\rho(\nu-p-1)}{\mu\beta}\right)\log r\right\} \tag{4}$$

Since $\frac{\rho}{\beta} < p + \frac{\rho-p}{\beta} < p + \frac{\nu-p}{\beta} < p + 1 + \frac{\rho(\nu-p-1)}{\mu\beta}$

we obtain from (2), (3) and (4) that for an infinity of r

$$\log M(r) < B \exp\left[\left\{p + 1 + \frac{\rho(\nu-p-1)}{\mu\beta}\right\}\log r\right]$$

Since $\mu - \lambda_1$ and $\nu - \rho$ may be chosen arbitrarily small we get

$$\lambda \leqslant p + 1 + \frac{\rho(\rho - p - 1)}{\lambda_1(\rho - p) + \rho(p + 1 - \rho)}$$

$$= p + \frac{(\rho - p)\lambda_1}{\lambda_1(\rho - p) + \rho(p + 1 - \rho)}$$

and the theorem is proved.

§3. **Corollary.** *Given* $\rho = \lambda$ and $\rho > 0$ *non-integer then* $\rho = \rho_1 = \lambda = \lambda_1$

For
$$\rho = \lambda \leqslant p + \frac{(\rho - p)\lambda_1}{\lambda_1(\rho - p) + \rho(p + 1 - \rho)}$$

$$\therefore \quad (\rho - p) \leqslant \frac{(\rho - p)\lambda_1}{\lambda_1(\rho - p) + \rho(p + 1 - \rho)}$$

$$\therefore \quad (p + 1 - \rho)(\rho - \lambda_1) \leqslant 0 \qquad \therefore \quad \rho \leqslant \lambda_1$$

Hence $\rho = \lambda_1$.