ON THE STRESS-STRAIN VELOCITY RELATIONS IN EQUATIONS OF VISCOUS FLOW

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Received February 8, 1943

Most of the mathematical results obtained for the motion of a viscous liquid are subject to serious limitations. More often than not we neglect the inertia terms. But even when it is possible not to neglect such terms the ordinary theory does not give satisfactory results in the case of many liquids we have to deal with in everyday life. In fact non-Newtonian liquids, *i.e.*, those in which the rate of shear is not proportional to the shearing stress do not obey Poiseinlli's or Stoke's law. It may therefore be of some interest to work out the consequences of a change in the stress-strain velocity relations.

Any method adopted to include in these relations second order terms must satisfy the following conditions:

(i) It must preserve their tensor form. For the sake of simplicity we shall assume it to be linear.

(ii) Like the equations of motion the strain velocities should be referred to a point (x, y, z) in the strained (actual) state of the liquid.

These conditions are satisfied if we take

$$\left.\begin{aligned} p_{xx} &= -p - \tfrac{2}{3}\mu\delta + 2\mu S_x, \\ p_{yz} &= \mu\sigma_{yz}, \text{ etc.} \end{aligned}\right\} \qquad (1)$$

where

$$p = -\tfrac{1}{3}(p_{xx} + p_{yy} + p_{zz}),$$

$$\delta = S_x + S_y + S_z,$$

$$S_x = \frac{\partial u}{\partial x} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right], \qquad (2\cdot 1)$$

$$\sigma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \left[\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right], \qquad (2\cdot 2)$$

with similar expressions for S_y, S_z, σ_{zx}, σ_{xy}.[1]

[1] *Cf.* Seth, *Phil. Trans. Roy. Soc.*, A, 1935, **234**, 231–64.

We cannot now use the ordinary equations of motion for a liquid given by

$$\rho \frac{D}{Dt}(u, v, w) = \rho (X, Y, Z) - \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) p + \mu \nabla^2 (u, v, w).$$

The fundamental equations of the type

$$\rho \frac{Du}{Dt} = \rho X + \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z} \qquad (3)$$

will have to be used.

We shall apply the above theory to the following particular cases of the steady motion of a liquid. The results obtained may not have much quantitative significance, but their qualitative value is not devoid of interest.

(i) Liquid between two parallel fixed planes.

(ii) Flow of a liquid through a pipe.

(iii) Two-dimensional simple rotatory motion.

Liquid between Two Fixed Parallel Planes

Let the origin be taken in one of these planes, and the axis of z perpendicular to them. The tentative assumption

$$u = f(z), \; v = 0, \; w = 0,$$

satisfies the equation of continuity and gives

$$p_{xx} = p_{yy} = -p + \tfrac{1}{3}\mu f'^2, \; p_{zz} = -p - \tfrac{2}{3}\mu f'^2,$$

$$p_{yz} = \mu f', \; p_{zx} = 0, \; p_{xy} = 0,$$

where $f' = df/dz$.

The equations of motion give

$$\frac{\partial p}{\partial x} = \mu \frac{d^2 f}{dz^2}, \; \frac{\partial p}{\partial y} = 0, \; \frac{\partial p}{\partial z} = -\tfrac{4}{3} u \frac{df}{dz}\frac{d^2 f}{dz^2},$$

which shew that

$$\frac{d^2 f}{dz^2} = 2A,$$

$$u = f(z) = Az^2 + Bz + C,$$

$$p = 2\mu Ax - \tfrac{2}{3}\mu (2Az + B)^2 + K,$$

where A, B, C and K are all constants.

The condition of no slipping at the boundary gives $C = 0$, $B + Ah = 0$, h being the distance between the fixed planes. Then

$$u = Az(z-h), \tag{4}$$

$$p = 2\mu Ax - \tfrac{2}{3}\mu A^2 (2z-h)^2 + K. \tag{5}$$

If $p = p_1$ at $x = 0$, $z = \frac{1}{2}h$, and $p = p_2$ at $x = l$, $z = \frac{1}{2}h$, we get

$$A = -(p_1 - p_2)/2\mu l, \ K = p_1.$$

Again, if u_0 is the mean velocity per unit breadth across a plane perpendicular to x, we have

$$u_0 = \frac{p_1 - p_2}{12\mu l} h^2. \tag{6}$$

p can now be re-written as

$$p = p_1 - \frac{p_1 - p_2}{l} x - \frac{1}{6} \frac{(p_1 - p_2)^2}{l^2 \mu} (2z-h)^2$$

$$= p_1 - \frac{12\mu u_0}{h^2} x - \frac{24\mu u_0^2}{h^4}(2z-h)^2. \tag{7}$$

The minimum value of p, say p_m, occurs when $x = l$ and $z = 0$ or h.

Thus

$$p_m = p_2 - \frac{24\mu}{h^2} u_0^2,$$

which is negative unless

$$u_0 < \tfrac{1}{2} h \left(\frac{p_2}{6\mu}\right)^{\frac{1}{2}}, \tag{8·1}$$

or

$$\frac{h}{l} < \frac{1}{p_1 - p_2} (6\mu p_2)^{\frac{1}{2}}. \tag{8·2}$$

No such result is given by the ordinary theory.

To get a numerical idea of the critical value of u_0 we put in C.G.S. units

$$p_2 = 980 \times 76 \times 13{\cdot}6, \ h = 0{\cdot}1, \ \mu = 0{\cdot}01,$$

and we get

$u_0 = 205{\cdot}3$ (approx.), the corresponding Reynold's number being 2053.

Flow of a Liquid through a Circular Pipe

Taking the axis of pipe as the z-axis we put

$$u = 0, \ v = 0, \ w = f(r), \ r^2 = x^2 + y^2,$$

which give

$$p_{xx} = -p + \tfrac{1}{3}\mu f'^2 - \frac{\mu x^2}{r^2} f'^2,$$

$$p_{yy} = -p + \tfrac{1}{2}\mu f'^2 - \frac{\mu y^2}{r^2} f'^2,$$

$$p_{zz} = -p + \tfrac{1}{2}\mu f'^2,$$

$$p_{yz} = -\mu \frac{yf'}{r}, \ p_{zx} = \frac{\mu x f'}{r}, \ p_{xy} = -\frac{\mu xy f'^2}{r^2}$$

when $f' = df/dr$.

The equations of motion give, assuming that gravity is the only external force,

$$\frac{\partial p}{\partial x}+\mu\left[\frac{2}{3r}\frac{df'^2}{dr}+\frac{f'^2}{r}\right]x=0,$$

$$\frac{\partial p}{\partial y}+\mu\left[\frac{2}{3r}\frac{df'^2}{dr}+\frac{f'^2}{r}\right]y=0,$$

$$\frac{\partial p}{\partial z}-\frac{\mu}{r}\frac{d}{dr}(rf')-\rho g=0.$$

These give

$$w=f=\tfrac{1}{4}\mathrm{A}r^2+\mathrm{B}\log r+\mathrm{C},$$

$$p=(\mathrm{A}\mu+\rho g)\,z-\frac{7\mu}{24}\mathrm{A}^2r^2+\mathrm{D}.$$

Let p_1, p_2 be the values of p at the ends of the axis of the pipe whose length is l. The condition of no slipping at the boundary and that w cannot be infinite for $r=0$ give

$$\mathrm{B}=0,\ \mathrm{C}=-\tfrac{1}{4}a^2,\ \partial=p_1,\ \mathrm{A}\mu+\rho g=-(p_1-p_2)/l.$$

Thus

$$w=\frac{1}{4\mu}\left[\frac{p_1-p_2}{l}+\rho g\right](a^2-r^2), \tag{9}$$

from which the total flow across a section is found to be

$$\int_0^a 2\pi r.\,w dr=\frac{\pi a^4}{8\mu}\cdot\left[\frac{p_1-p_2}{l}+\rho g,\right] \tag{10}$$

and

$$p=p_1-\frac{p_1-p_2}{l}z-\frac{7}{24\mu}\left[\frac{p_1-p_2}{l}+\rho g\right]^2r^2. \tag{11}$$

Unlike the ordinary theory p is not constant over a cross-section, the variation being given by the last term in (11).

Ii w_0 is the mean velocity over a section, and p_m the minimum value of p which occurs at $z=l$ and $r=a$

$$w_0=\frac{a^2}{8\mu}\left[\frac{p_1-p_2}{l}+\rho g,\right], \tag{12}$$

$$p_m=p_2-\frac{56}{3}\mu\frac{w_0^2}{a^2}, \tag{13}$$

which will be negative unless

$$w_0<\tfrac{1}{2}a\left(\frac{3p_2}{14\mu}\right)^{\frac{1}{2}}. \tag{14}$$

If in C.G.S. units we assume that

$$p_2 = 980 \times 76 \times 13\cdot 6,\ a = 0\cdot 1,\ \mu = 0\cdot 018,$$

the critical value of the mean velocity is given by

$$w_0 = 173\cdot 5 \text{ (approx.)}.$$

the corresponding value of the Reynold's number being 1928.

Flow through a Pipe with an Elliptic Section

In this case we can take

$$u = 0,\ v = 0,\ w = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

and proceeding as above we find

$$p = p_1 - 4\mu \frac{a^2 + b^2}{a^2 b^2} w_0 z - \tfrac{8}{3}\mu w_0^2 \left[\frac{x^2}{a^2}\left(\frac{4}{a^2} + \frac{3}{b^2}\right) + \frac{y^2}{b^2}\left(\frac{4}{b^2} + \frac{3}{a^2}\right)\right], \tag{15}$$

whose minimum value is

$$p_m = p_2 - \tfrac{8}{3}\mu w_0^2 \left(\frac{4}{b^2} + \frac{3}{a^2}\right),\ a > b$$

which is negative unless

$$w_0 < \tfrac{1}{2} ab \left[\frac{3p_2}{2\mu(4a^2 + 3b^2)}\right]^{\frac{1}{2}}. \tag{16}$$

Neglecting powers of e^2 higher than the second we get

$$w_0 < \tfrac{1}{2} a \left(\frac{3p_2}{14\mu}\right)^{\frac{1}{2}} \left(1 - \frac{4}{7} e^2\right), \tag{17}$$

which shews that the limit is decreased as compared with a circular pipe of radius a.

The total flow across a section remains unchanged to this order of approximation.

Two Dimentional Rotatory Motion

Assuming the axis of rotation as the z-axis we can put

$$u = -yf(r),\ v = xf(r),\ w = 0,\ r^2 = x^2 + y^2,$$

which give

$$p_{xx} = -p + \tfrac{1}{3}\mu(r^2 f'^2 + 2r ff' + 2f^2)$$
$$- 2\mu\left[\frac{f'xy}{r} + \tfrac{1}{2}\left(x^2 f'^2 + \frac{2x^2 ff'}{r} + f^2\right)\right],$$

$$p_{yy} = -p + \tfrac{1}{3}\mu\,(r^2 f'^2 + 2r ff' + 2f^2)$$
$$\qquad + 2\mu\left[\frac{f'xy}{r} - \tfrac{1}{2}\left(y^2 f'^2 + \frac{2y^2 ff'}{r} + f^2\right)\right],$$
$$p_{zz} = -p + \tfrac{1}{3}\mu\,(r^2 f'^2 + 2rff' + 2f^2),$$
$$p_{yz} = 0,\ p_{zx} = 0,\ p_{xy} = \mu\left[\frac{f'}{r}(x^2 - y^2) - \frac{xy}{r} f'(2f + rf')\right]$$

The equations of motion are all found to be satisfied if

$$\frac{\partial p}{\partial z} = 0,\ r\frac{\partial}{\partial r}\left(\frac{f'}{r}\right) + \frac{4f'}{r} = 0;$$
$$\frac{dp}{dr} = \rho r f^2 + \tfrac{1}{3}\mu\frac{\partial}{\partial r}(r^2 f'^2 + 2rff' + 2f^2)$$
$$\qquad - \mu\left[3rf^2 + 8ff' + r^2\frac{\partial}{\partial r}\left(\frac{2ff'}{r} + f'^2\right)\right],$$

which give

$$f = \tfrac{1}{2}\frac{A}{r^2} + B,$$

and, if the liquid is at rest at infinity and the internal boundary is a solid cylinder of radius a, $B = 0$ and

$$f = \frac{a^2 w_0}{r^2},$$

w_0 being the angular velocity of the cylinder.

The frictional couple on the cylinder is found to be

$$\left(\mu r\frac{df}{dr}\cdot 2\pi r^2\right)_{r=a} = -4\pi\mu a^2 w_0,$$

and the value of p as

$$p = \Pi - \tfrac{1}{2}\rho\frac{a^4 w_0^2}{r^2} - \tfrac{4}{3}\mu\frac{a^4 w_0^2}{r^4}, \tag{18}$$

whose minimum value is

$$p_m = \Pi - \tfrac{1}{6} w_0^2 (3\rho a^2 + 2\mu),$$

Π being the mean pressure at infinity.

p_m is negative unless

$$w_0 < \left(\frac{6\Pi}{3\rho a^2 + 2\mu}\right)^{\frac{1}{2}}. \tag{19}$$

No such limit to the value of w_0 is given by the ordinary theory.

Liquid Rotating between Two Co-Axial Cylinders

If the liquid is bounded externally by a fixed co-axial cylinder of **radius** b, we get from the above

$$f=\frac{a^2}{r^2}\cdot\frac{b^2-r^2}{b^2-a^2}\,w_0, \tag{20}$$

$$p=p_0-\tfrac{1}{3}\mu\frac{a^4w_0^2}{(b^2-a^2)^2}\left(\frac{b^4}{r^4}-1\right)$$

$$-\rho w_0^2\left(\frac{a^2}{b^2-a^2}\right)^2\left[\tfrac{1}{2}r^2\left(\frac{b^4}{r^4}-1\right)+2b^2\log\frac{r}{b}\right], \tag{21}$$

$$p_m=p_0-\tfrac{1}{3}\mu w_0^2\frac{b^2+a^2}{b^2-a^2}-$$

$$-\rho w_0^2\frac{a^2}{(b^2-a^2)^2}\left[\tfrac{1}{2}(b^4-a^4)+2a^2b^2\log\frac{a}{b}\right], \tag{22}$$

where $p=p_0$ over $r=b$.

The critical value of w_0 is obtained by equating p_m to zero.

If, on the other hand, the inner is fixed we find from (22) that p_m is always positive, and hence the motion is always possible. This result is interesting in view of what Taylor[2] has discovered in connection with the rotation of a liquid between two co-axial cylinders. He finds that when the inner cylinder is fixed the steady motion is stable for all observed speeds of rotation of the outer one. When the outer is fixed there is stability only for sufficiently low speeds of the inner one.

Summary

Three simple cases of the steady motion of a viscous liquid are discussed by assuming a general form of the stress-strain velocity relations so as to include second order terms in them. The conditions to be satisfied by these relations are that they must preserve their tensor form assumed linear for simplicity, and that the strain velocities should be referred to a point in the strained (actual) state of the liquid. The results obtained are compared with those of the ordinary theory.

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[2] G. I. Taylor, *Bhil. Trans. Roy. Soc.*, A, 1922, **289**, 289-343.