SOME APPLICATIONS OF RAMANUJAN'S TRIGONOMETRICAL SUM $C_m(n)$

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The object of this paper is to exhibit the application of Ramanujan's trigonometrical sum to two arithmetical theories, the theory of Relative partitions (mod $m$) of Von Sterneck and the theory of the class division of the integers mod $m$ of Dr. R. Vaidyanathaswamy.

If $n = e_1 + e_2 + \ldots \pmod{m}$, $n$ is said to be relatively partitioned (mod $m$). Von Sterneck obtained explicit expressions for various numerical functions in this theory. He showed that these functions assume neat forms when expressed in terms of a certain arithmetic function $f(n, m)$ of two arguments. This function of Von Sterneck was recently proved by me to be identical with Ramanujan's trigonometrical sum $C_m(n)$. Using this fact I prove all of Von Sterneck's results by a method which besides being easy and direct shows clearly the fundamental nature of the trigonometrical sum in this theory.

Dr. R. Vaidyanathaswamy studied a class division of the integers mod $m$, in which these integers are divided into a certain number of classes $C_1, C_2, \ldots$ according to their g.c.d. with $m$. He proved the remarkable theorem that these classes combine by addition, i.e., that they form elements of a linear associative algebra with the scheme

$$C_i \cdot C_j = \sum_k \gamma^k_{ij} C_k$$

where $C_i \cdot C_j$ means the set of numbers, obtained by adding each number of $C_i$ to each number of $C_j$. It is shown here that $\gamma^k_{ij}$ could be expressed in terms of Ramanujan's sum. In fact

$$\gamma^k_{ij} = \frac{1}{m} \sum_{\delta|m} \frac{C_m(\delta)}{i_1} C_m(\delta) \frac{C_m(\delta)}{i_2} C_m(\delta)$$

* I am indebted to Dr. R. Vaidyanathaswamy for his help in the preparation of this paper.

1 Collected papers of S. Ramanujan (Cambridge), 1927, p. 179.
2 Bachmann, Niedere Zahlentheorie, Bd. 2, 222-41.
These references will hereafter be quoted by the numbers given above.
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I prove more generally that

**Theorem.**—$C_1 \alpha_1 + C_2 \alpha_2 + \ldots = \sum_k A_k C_k \quad \text{where}$

$$A_k = \frac{1}{m} \sum_{\delta | m} C_{\frac{m}{\delta}} \left( \alpha_1 \right) C_{\frac{m}{\delta}} \left( \alpha_2 \right) \ldots C_{\frac{m}{\delta}} \left( \alpha_n \right).$$

I find also an expression for a certain numerical function connected with the theory of relative partitions (mod $m$), the set of integers used being those less than and prime to $m$. I prove also the interesting results.

1. If $m$ is even then every odd number is the sum mod $m$ of three and every even number is the sum mod $m$ of two numbers less than and prime to $m$.
2. If $m$ is odd then every number is the sum mod $m$ of two numbers less than and prime to $m$.

2. Ramanujan's sum is

$$C_m(n) = \sum_k e^{2\pi i k n / m} = C_m(-n),$$

where $k$ runs through all the integers less than and prime to $m$. Hardy proved that

$$C_m(n) \cdot C_m(n') = C_{mm'}(n) \quad (m, m') = 1$$

and

$$C_m(n) = \sum \mu \left( \frac{m}{\delta} \right) \delta$$

the summation being over the common divisors of $m$ and $n$ and $\mu(n)$ is the Moebius function. We shall now prove the

**Lemma A*.—**

$$C_m(n) = \mu \left( \frac{m}{d} \right) \frac{\phi(m)}{\phi \left( \frac{m}{d} \right)}, \quad d = (m, n)$$

where $\phi(n)$ is Euler's function.

**Proof.**—

$$C_m(n) = \sum_{\delta | d} \mu \left( \frac{m}{\delta} \right) \delta = \sum_{\delta | d} \mu \left( \frac{m}{d} \cdot \delta \right) \delta.$$ 

We might sum, naturally, for those divisors $d_1$ of $d$ which are prime to $\frac{m}{d}$, for otherwise $\mu \left( \frac{m}{d} \cdot \delta \right)$ vanishes. Thus

$$C_m(n) = d \sum_{d_1} \mu \left( \frac{m}{d} \right) \mu \left( d_1 \right) d_1^{-1} \quad = d \mu \left( \frac{m}{d} \right) \sum_{d_1} \mu \left( d_1 \right) d_1^{-1}$$

which is the right-hand side of the lemma.

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* Hardy and Wright, Introduction to the Theory of Numbers, p. 231.
* There is another proof in my paper, Ref. 4. See also Õ. Holder. Prace. Matematyczno-Fizyczne, 1936, 13-23.
Corollary.—$C_m(n)$ depends on $n$ only through its g.c.d. with $m$ so that $C_m(n) = C_m(d)$.

Lemma B.—If $f(m, r)$ and $\phi(m, r)$ be two arithmetic functions possessing the modulus $m$ and if
\[
\sum_{k=0}^{m-1} f(m, k) \rho^{kr} = \phi(m, r) \quad \rho = e^{\frac{2\pi i}{m}}
\]
then
\[
\sum_{k=0}^{m-1} \phi(m, k) \rho^{-kr} = m f(m, r)
\]
Proof.—
\[
\sum_{k=0}^{m-1} \phi(m, k) \rho^{-kr} = \sum_{\lambda=0}^{m-1} \sum_{k=0}^{m-1} f(m, k) \rho^{(k-r)\lambda}
\]
The inner sum is zero except when $k = r$ when its value is $m$.

Lemma C.—If $\phi(m, r)$ depends on $r$ only through its g.c.d. with $m$ then so does $f(m, r)$ and then each of them can be expressed in terms of the other and Ramanujan's sum.
Proof.—
\[
\sum_{k=0}^{m-1} f(m, k) = \sum_{\lambda=0}^{m-1} \sum_{k=0}^{m-1} \rho^{(k-r)\lambda}
\]
so that
\[
m f(m, k) = \sum_{\delta|m} \phi(m, \delta) C_m(k)\]
and
\[
\phi(m, k) = \sum_{\delta|m} f(m, \delta) C_m(k).
\]

3. We shall now prove Von Sterneck's results by using the above lemmas.

**Theorem I.** $C_m(n) = \sum_{k=0}^{m-1} (-1)^k (n)_k^{(0)} = \sum_{\nu} (-1)^\nu$

where $(n)_k^{(0)}$ is the number of ways of expressing $n$ as the sum mod $m$ of $k$ different elements of the set $1, 2, 3, \ldots, (m - 1)$ and $\nu$ is the number of parts in a relative partition of $n$ mod $m$ into distinct parts not including zero.

Proof.—If $\rho = e^{\frac{2\pi i}{m}}$ then
\[
(1 - \rho^r)(1 - \rho^{2r}) \ldots (1 - \rho^{(m-1)r}) = \sum_{k=0}^{m-1} f(m, k) \rho^{rk}
\]
where
\[
f(m, k) = \sum_{\nu} (-1)^\nu (k)_\nu^{(0)}
\]
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But $(1 - \rho^r)(1 - \rho^{2r})\ldots(1 - \rho^{(m-1)r}) = 0 \quad (m, r) > 1$

so that using lemma (c) we have

$m f(m, k) = \sum_r m \rho^{-rk} \quad (r, m) = 1 \quad 1 \leq r < m$

$m C_m(k)$.

This is a direct and simple proof of the identity of Ramanujan's sum and Von Sterneck's function.

**Theorem 2.**

$$A(m, n) = \sum_{i=0}^{m-1} (n)(\sum_{\lambda=1}^{2m} \phi(m, \lambda))$$

*summation being for all odd divisors of m.*

*Proof.* It is easy to see that

$$\sum_{k=0}^{m-1} A(m, k) \rho^{kr} = \prod_{\lambda=1}^{2m} (1 + \rho^{kr}) = \frac{1}{\phi(m, r)}$$

where $\phi(m, r) = \prod_{\lambda=1}^{2m} (1 + \rho^{kr})$.

The value of $\phi(m, r)$ depends on $r$ only through its g.c.d. with $m$ so that using lemma C we have

$$m A(m, n) = \frac{1}{\phi(m, \delta)} \sum_{\delta|m} \phi(m, \delta) C_m(n).$$

But $\phi\left(m, \frac{m}{\delta}\right) = \left\{1 + e\left(\frac{1}{\delta}\right)\right\} \left\{1 + e\left(\frac{2}{\delta}\right)\right\} \ldots \left\{1 + e\left(\frac{\delta-1}{\delta}\right)\right\}^{\frac{m}{\delta}}$

where $e(x) = e^{2\pi i x}$.

Since

$$\frac{\sin m \theta}{\sin \theta} = 2^{m-1} \sin(\theta + \beta) \ldots \sin(\theta + \frac{m-1}{m} \beta)$$

where $\beta = \frac{\pi}{m}$

we see that, by putting $\theta = \frac{\pi}{2}$.

$$\left[1 + e\left(\frac{1}{\delta}\right)\right] \ldots \left[1 + e\left(\frac{\delta-1}{\delta}\right)\right] = \sin \frac{\delta m}{2} \times (-1)^{\frac{\delta - 1}{2}}.$$ 

Substituting this value we have the required result.

**Theorem 3.**— *If $(n)_k$ denotes the number of ways of expressing $n$ as the sum (mod $m$) of $k$ integers of the set 0, 1, 2, $\ldots$ $m - 1$ repetitions being allowed then*

$$[n]_k = \frac{1}{m} \left\{\sum_{\delta|m} \left(\frac{m + k}{\delta} - 1\right)\right\} C_\delta(n)$$

*where $\left(\frac{m}{n}\right)$ is the usual coefficient which vanishes if $n$ or $m$ is non integral.*
Proof.—It is easily seen that if \( n \leq m \) and
\[
A_a (r) = \sum_{k=0}^{\frac{2m}{m}} [k_a] \rho^k
\]
then \( A_a (r) \) is the coefficient of \( x^a \) in the expansion of \([(1 - x\rho^r)(1 - x\rho^{2r}) \ldots (1 - x\rho^{mr})]^{-1} \) as a power series in \( x \). But \((1 - x\rho^r) \ldots (1 - x\rho^{mr}) = (1 - x^{m\delta})^\delta \) where \( \delta = (m, r) \) so that \( A_a (r) \) is the coefficient of \( x^a \) in the binomial expansion of \((1 - x^{m\delta})^{-\delta}\)

\[\therefore A_a (r) = 0 \text{ if } m/\delta \text{ does not divide } a \]
\[= (-1)^\lambda \left( -\frac{\delta}{\lambda} \right) \text{ when } \lambda = \frac{a\delta}{m} \]
Thus \( A_k (r) = \sum_{n=0}^{\frac{m-1}{\delta}} [n] k \rho^{nr} \) and by lemma C we have the result.

Theorem 4.—If \((n)_k\) denotes the number of ways of expressing \( n \) as the sum \((\text{mod } m)\) of \( k \) distinct integers of the set 0, 1, \ldots, \( m - 1 \) then
\[
(n)_k = \sum_{\delta|m} (-1)^{\lambda} \left( \frac{m/\delta}{k/\delta} \right) C_\delta (n)
\]
Proof.—As in the previous theorem we see easily that \( B_a (r) = \sum_{k=0}^{\frac{m-1}{\delta}} [k_a] \rho^k \rho^r \) is the coefficient of \( x^a \) in the expansion of \((1 + x\rho^r)(1 + x\rho^{2r}) \ldots (1 + x\rho^{mr}) \).

But \((1 - x\rho^r)(1 - x\rho^{2r}) \ldots (1 - x\rho^{mr}) = (1 - x^{m\delta})^\delta \).

\[\therefore B_a (r) = 0 \text{ if } m/\delta \text{ does not divide } a.\]
\[= (-1)^\lambda \left( -\frac{\delta}{\lambda} \right) \times (-1)^{m\lambda/\delta} \text{ when } \lambda = \frac{a\delta}{m} \]
\[= (-1)^{\lambda \frac{m\lambda}{\delta} + \lambda \left( \frac{\delta}{\lambda} \right)} \]
\[\therefore B_k (r) = \sum_{n=0}^{\frac{m-1}{\delta}} (n)_k \rho^{nr}.\]
By lemma C we have the theorem.

4. We now proceed to the class division of the integers \( \text{mod } m \). Let \( t_1 (= 1), t_2, \ldots, t_\lambda (= m) \) the number of divisors of \( m \) be the distinct divisors of \( m \). Dr. R. Vaidyanathaswamy divides the integers 1, 2, \ldots, \( m \) into \( \lambda \) classes \( C_1, C_2, \ldots, C_\lambda \) in such a way that \( C_r \) contains those integers \( \text{mod } m \) which have with \( m \) a g.c.d. equal to \( t_r \). Thus the number of elements in any set \( C_r \) is \( \phi \left( \frac{m}{t_r} \right) \). These classes combine among themselves by means of addition. Let \( C_r \) consist of the integers \( \beta_1, \beta_2, \ldots, \beta_{g_r} \) where \( g_r = \phi \left( \frac{m}{t_r} \right) \). We shall prove the following:
THEOREM 5.

\[ C_i C_j = \sum_k \gamma_{ij}^k C_k \]

where

\[ \gamma_{ij}^k = \frac{1}{m} \sum_{t_i} C_m(\delta) C_m(\delta) C_m(t_k). \]

Proof.—If \( \rho = e^{\frac{2\pi i}{m}} \) and

\[ (\rho^{r_{11}} + \rho^{r_{21}} + \ldots + \rho^{r_{ij}}) (\rho^{r_{12}} + \rho^{r_{22}} + \ldots + \rho^{r_{ij}}) \]

then \( f(m, n) \rho^{nr} \)

\[ = \sum_{n=0}^{m-1} f(m, n) \rho^{nr} \]

then \( f(m, n) \) is the number of ways of expressing \( n \) as the sum (mod \( m \)) of two numbers one from each of the sets \( C_i \) and \( C_j \).

It is easy to see that \( \sum \rho^{nr} \) where \( \alpha \) runs through all the elements of the set \( C_k \) has the value \( C_m(r) \). Thus

\[ \sum_{n=0}^{m-1} f(m, n) \rho^{nr} = C_m(r) C_m(r). \]

By lemma C we have the result.

THEOREM 6.

\[ C_{a_1} C_{a_2} \ldots = \sum_k A_k C_k \]

where

\[ A_k = \frac{1}{m} \sum_{t_i} C_{a_1}(\delta) \ldots C_{a_k}(t_k). \]

Proof.—If \( f(m, n) \) represents the number of ways of expressing \( n \) as the sum (mod \( m \)) of \( a_1 \) numbers of the set \( C_1 \), \( a_2 \) numbers of the set \( C_2 \) and so on then

\[ \phi(m, r) = \sum_{n=0}^{m-1} f(m, n) \rho^{nr} = \prod_{i=1}^{a_i} \left( \sum_{s=1}^{r_i} \rho^{r_i s} \right)^{a_i}. \]

By an argument similar to the one used in the previous theorem we have the result.

THEOREM 7.

\[ C_i = \sum_k A_k C_k \]

where

\[ A_k = \frac{\phi^r(m)}{m} \prod_{\rho \mid m \rho \neq 1} \left( \frac{(p-1)^r - (-1)^r}{(p-1)^r} \right) \prod_{\rho \mid t_k} \left( \frac{(p-1)^r - (-1)^r}{(p-1)^r-1} \right) \]

\( p \) being a prime number, \( \phi^r(m) = [\phi(m)]^r \) and \( \phi(m) \) is Euler's totient function.

Proof.—As before if \( f(m, n) \) is the number of representations of \( n \) as the sum (mod \( m \)) of \( r \) integers of the set \( C_i \) then we have easily

\[ \sum_{n=0}^{m-1} f(m, n) \rho^{nr} = C_m(\lambda). \]
and by the usual inversion (lemma C) we have

\[ f(m, n) = \frac{1}{m} \sum_{\delta|m} \phi^r(m/\delta) C_m(\delta) C_m(n). \]

By lemma A we have

\[ f(m, n) = \frac{\phi^r(m)}{m} \sum_{\delta|m} \frac{\mu^r(\delta)}{\phi^r(\delta)} C_\delta(n) \]

\[ = \frac{\phi^r(m)}{m} \prod_{p|m} \left[ 1 + \frac{(-1)^r}{\phi^r(p)} C_{\phi(p)}(n) \right] \]

But \( C_{\phi(p)}(n) = -1 \) if \( p \nmid n \) and \( = p - 1 \) if \( p \mid n \). Using this we have the required result.

**Corollaries.**—(1) If \( m \) is even then every odd number is the sum \((\text{mod } m)\) of 3 and every even number is the sum \((\text{mod } m)\) of 2 numbers less than and prime to \( m \).

(2) If \( m \) is odd then every number is the sum \((\text{mod } m)\) of two numbers less than and prime to \( m \).

These follow easily from the above theorem because we have merely to find the least \( r \) for which no \( A_k \) is zero when \( m \) is even all \( A_k \)'s, for which the corresponding \( t_k \)'s are even, are zeroes. When \( m \) is odd, \( r = 2 \), no \( A_k \) is zero.

\[ \sum_{k=1}^{m} C^r_m(k) = \phi^r(m) \prod_{p|m} \left[ 1 - \frac{(-1)^{r+1}}{(p-1)^r-1} \right]. \]

This follows easily from the result \( \sum_{n=0}^{m-1} f(m, n) \rho^m = C_m^{\lambda}(\lambda) \) by using lemma A and putting \( n = m \).

\[ \sum_{k=1}^{m} C^2_m(k) = m\phi(m). \]

5. We shall study the problem similar to that considered by Von Sterneck but confining ourselves to the integers less than and prime to \( m \).

**Theorem 8.**—If \( f(m, n) \) denotes the excess of the number of relative partitions of \( n \) \((\text{mod } m)\) into an even number of parts over those into an odd number, the parts being all distinct and chosen from the set of integers less than and prime to \( m \) then

\[ f(m, n) = \frac{1}{m} \sum \text{Exp.} \left( \frac{\wedge \left( \frac{m}{\delta} \right) \phi(m)}{\phi \left( \frac{m}{\delta} \right)} \right) C_m(\delta) \]

where \( \text{Exp.} (x) \) means \( e \) and \( \wedge (n) \) is the arithmetic function defined by

\[ -\frac{d}{ds} \log \zeta(s) = \sum_{n=1}^{\infty} \frac{\triangle(n)}{n^s}. \]

\( \zeta(s) \) being Riemann zeta function.
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Proof.—Using the notation of Section 3 we have

$$(1 - \rho^{\beta_1 r}) (1 - \rho^{\beta_2 r}) \ldots (1 - \rho^{\beta_\lambda r}) = \sum_{k=0}^{\frac{m-1}{d}} f(m, k) \rho^{kr}.$$ 

But it is known$^4$ that

$$(1 - \rho^{\beta_1}) (1 - \rho^{\beta_2}) \ldots (1 - \rho^{\beta_\lambda}) = e^{\lambda(m)}$$

and therefore

$$\sum_{k=0}^{\frac{m-1}{d}} f(m, k) \rho^{kr} = \exp\left[\lambda\left(\frac{m}{d}\right) \phi(m)\right].$$

using lemma C we have the required result.

6. It is well known$^7$ that if $\rho_1, \rho_2, \ldots, \rho_\lambda$ $\lambda = \phi(m)$ be the primitive $m$th roots of unity then

$$(x - \rho_1) (x - \rho_2) \ldots (x - \rho_\lambda) = \prod_{d|m} (x^{\frac{d}{m}} - 1)^{\mu(m)}.$$ 

It is known from Newton's theorem that the coefficients in the product could be expressed in terms of the sums of powers of roots. But

$$\rho_1^{k_1} + \rho_2^{k_2} + \ldots + \rho_\lambda^{k_\lambda} = C_m(k)$$

and thus we get

$$\pi \left( \sum_{d|m} (x^{\frac{d}{m}} - 1)^{\mu(m)} \right) = \sum_{r=0}^{\lambda} A_r x^r$$

where

$$A_r = \left(\frac{-1}{r!}\right)^{1/r} \left| \begin{array}{ccc}
C_m(1) & 1 & 0 \\
C_m(2) & C_m(1) & 2 \\
& & \\
& & \\
& & \\
& & \\
C_m(r-1), C_m(r-2) & \ldots & C_m(1) (r-1) \\
C_m(r), C_m(r-1) & \ldots & C_m(1) \\
\end{array} \right|$$

which shows that if $m$ has a square factor (at least) prime to $r$! $A_r$ and all the previous ones therefore vanish. It is also of interest to notice that since

$$(1 - \rho_1) (1 - \rho_2) \ldots (1 - \rho_\lambda) = e^{\lambda(m)}$$

we have

$$A_0 + A_1 + A_2 + \ldots + A_\lambda = e^{\lambda(m)}$$

which shows that a cyclotomic equation (Kreisteilungsgleichung) of degree $\phi(m)$ has the sum of the positive coefficients greater than the negative ones always. since $e^{\lambda(m)}$ is never negative.

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