THE LOCALISATION THEORY IN SET-TOPOLOGY

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The earliest notion which could be described as the starting point of set-topology is Cantor’s concept of the derived set $D(X)$ of a set $X$ of real numbers, defined as the set of points of $X$ which are accumulation points thereof. This was generalised by Frechet into the abstract topological notion of the derived set $D(X)$ of a set $X$, defined as the set of all limit-elements of $X$. In the present-day treatments of set-topology, a greater prominence is generally given to the closure function $X = X + D(X)$, introduced postulationally by Kuratowski. The connection between the closure and the derivate of a set $X$ may be described by saying that in the topology defined by the closure-function, both the closure $X$ and the derived set $D(X)$ appear as local functions of $X$; namely $X$ is the set of points at which $X$ is not locally null, while (assuming that the topology is $T_1$) $D(X)$ is the set of points at which $X$ is not locally finite. The idea of localisation of properties thus suggested has been treated in a general manner by Kuratowski in his Topologie, with a systematic calculus; a notable achievement of the calculus is the elegant proof that it gives of the theorem that the points at which a set of the second category is locally of the second category constitute a closed domain. (The corresponding theorem for metric spaces was originally stated and proved by Banach.)

In this paper, I review the localisation theory and study the properties of what I have called compact and super-compact ideals (or hereditary additive properties), as well as certain extensions $P_a, P_s, P_N$ of an ideal $P$.

I. Let $R$ be a topological $T_1$-space, $B_R$ the boolean algebra of all its subsets. A property $P$ of subsets of $R$ is hereditary if $Y \subseteq X$ and $X \in P$ imply $Y \in P$; it is additive, if $X \in P$ and $Y \in P$ imply $X + Y \in P$. Given any property $P$ of subsets of $R$, it is convenient to denote also by $P$ the family of all subsets possessing the property $P$. If $P$ is a hereditary additive property, it is clear that the family $P$ is a $\mu$-ideal of $B_R$; conversely corresponding to any $\mu$-ideal $P$ of $B_R$ we have a hereditary additive property $P$, viz., the property of belonging to the ideal $P$. In particular if $P$ is the zero ideal, i.e., the ideal containing the null set only, the corresponding hereditary additive property is the property of being null; if $P$ is the ideal 1, i.e., the
whole of $B_R$, the corresponding property is the universal property of being a subset of $R$; and so on.

We say that a set $X$ has the property $P$ locally at a point $x$, if there exists a neighbourhood $U_x$ of $x$, such that $U_x \cdot X \in P$. The set of points at which $X$ does not have the property $P$ locally is denoted by $P(X)$. Thus $x \in P(X)$ means that for every neighbourhood $U_x$ of $x$, $U_x \cdot X$ does not have the property $P$, or briefly is not a $P$. If $P$ is a hereditary property it will be sufficient to restrict ourselves to open neighbourhoods; for if $U_x \cdot X \in P$, $(\text{Int. } U_x) \cdot X < U_x \cdot X$ is also a $P$, and since $\text{Int. } U_x$ is also neighbourhood, $(\text{Int. } U_x) \cdot X \in P$ implies $X$ is locally $P$ at $x$.

We shall consider only hereditary additive properties $P$; we shall call $P(X)$ the local function of the corresponding ideal $P$.

II. General Properties of the local function $P(X)$. *

(a) If $X < Y$, $P(X) < P(Y)$.

For if $p$ is not in $P(Y)$, it has an open neighbourhood $G_p$ such that $G_p \cdot Y \in P$. Then $G_p \cdot X < G_p \cdot Y$ is also a $P$. Hence $p$ is not in $P(X)$. Thus $\{P(Y)\}' < \{P(X)\}'$ or $P(X) < P(Y)$.

(b) If $P < Q$ (i.e., if the ideal $P$ is contained in the ideal $Q$), $P(X) > Q(X)$.

For if $X$ is locally $P$ at $x$, it must also be locally $Q$ at $x$ (since every $P$ is also a $Q$). Hence $\{P(X)\}' < \{Q(X)\}'$ or $P(X) > Q(X)$.

(c) $P(X)$ is a closed set contained in $\overline{X}$.

For $\overline{X}$ is the local function $0(X)$ of the zero ideal. Since the ideal $P$ contains the zero ideal, it follows from (b) that $P(X) < \overline{X}$. To shew that $P(X)$ is closed, any point $p \in \{P(X)\}'$ has an open neighbourhood $G_p$ such that $G_p \cdot X \in P$. It is clear that $X$ is locally $P$ at every point of $G_p$. Thus $G_p \cdot X < \{P(X)\}'$, or $\{P(X)\}'$ is open. Hence $P(X)$ is closed.

(d) $PP(X) < P(X)$.

For by (c) $PP(X) < P(X) = P(X)$.

(e) $P(X + Y) = P(X) + P(Y)$.

From (a) it follows that $P(X + Y) > P(X) + P(Y)$. To prove the reverse inclusion, let $p$ belong neither to $P(X)$ nor to $P(Y)$. Therefore it has open neighbourhoods $U_p$, $V_p$, so that $U_p \cdot X \in P$, $V_p \cdot Y \in P$. As $P$ is here-

*All these properties with the exception of (b) and (f) will be found in Kuratowski’s *Topologie*, pp. 29, 30.
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ditary and additive \( U_p V_p (X + Y) \in \mathcal{P} \). Since \( U_p V_p \) is an open neighbourhood of \( p \), it follows that \( p \) does not belong to \( \mathcal{P} (X + Y) \). Hence \( \{ \mathcal{P} (X) \}' \cdot \{ \mathcal{P} (Y) \}' \subset \{ \mathcal{P} (X + Y) \}' \) or \( \mathcal{P} (X + Y) \subset \mathcal{P} (X) + \mathcal{P} (Y) \). Hence the result.

(f) If \( \mathcal{P}_3 \) is the intersection of the ideals \( \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 (X) = \mathcal{P}_1 (X) + \mathcal{P}_2 (X) \).

For by (b) \( \mathcal{P}_3 (X) > \mathcal{P}_1 (X) + \mathcal{P}_2 (X) \). To prove the reverse inclusion, we observe that if \( X \) is locally \( \mathcal{P}_1 \) as well as locally \( \mathcal{P}_2 \) at \( x \), there must exist open neighbourhoods \( U_x, V_x \) of \( x \), so that \( U_x \cdot X \in \mathcal{P}_1, V_x \cdot X \in \mathcal{P}_2 \). Hence \( U_x V_x X \) is a \( \mathcal{P}_1 \) and also a \( \mathcal{P}_2 \); that is, \( U_x V_x X \in \mathcal{P}_3 \) or \( X \) is locally \( \mathcal{P}_3 \) at \( x \). Hence if \( X \) is not locally \( \mathcal{P}_3 \) at \( x \), it is either not locally \( \mathcal{P}_1 \) or not locally \( \mathcal{P}_2 \) there; \( i.e., \mathcal{P}_3 (X) < \mathcal{P}_1 (X) + \mathcal{P}_2 (X) \). This proves that \( \mathcal{P}_3 (X) = \mathcal{P}_1 (X) + \mathcal{P}_2 (X) \).

(g) \( \mathcal{P} (X) \{ \mathcal{P} (Y) \}' = \mathcal{P} (XY') \{ \mathcal{P} (Y) \}' < \mathcal{P} (XY') \).

This follows since \( \mathcal{P} (X) = \mathcal{P} (XY) + \mathcal{P} (XY') \) and \( \mathcal{P} (XY) < \mathcal{P} (Y) \) by (a).

(h) If \( G \) is open \( G \cdot \mathcal{P} (X) = G \cdot \mathcal{P} (GX) < \mathcal{P} (GX) \).

For if \( p \in G \cdot \mathcal{P} (X) \), and \( H \) any open neighbourhood of \( p \), \( HG \) is also a neighbourhood, and therefore \( HGX \) is not a \( \mathcal{P} \). Hence \( p \in \mathcal{P} (GX) \). Thus \( G \cdot \mathcal{P} (X) < \mathcal{P} (GX) < \mathcal{P} (X) \) by (a).

Hence \( GP (X) < G \cdot \mathcal{P} (GX) < GP (X) \), which proves (h).

The additive property of \( \mathcal{P} \) has been used in proving (e), (g) only.

Hence the remaining properties are all valid if \( \mathcal{P} \) is hereditary without being additive.

III. The fundamental series of ideals and their local functions.—It was already noticed that the closure function \( \overline{X} \) is the local function of the zero ideal \( \mathcal{O} \). This is the lowest of a series of ideals, which we may call numerical ideals, and denote by \( I_\omega \); where for any ordinal \( \omega > 0 \), we denote by \( I_\omega \) the ideal of all sets whose potency is less than \( N_\omega \). \( I_0 \) (which we may write simply \( I \)) is thus the ideal of finite sets, \( I_1 \) the ideal of sets which are either finite or enumerable and so on. \( \mathcal{O} \) and \( I (= I_0) \) are the basic ideals we have to consider; the local function \( \mathcal{O} (X) \) is \( X \), while the local function \( I (X) \) is the derived set of \( X \), defined as the set of all accumulation points of \( X \).† From II (c) \( I (X) \) is a closed set contained in \( X \), and \( \overline{X} = X + I (X) \). A set \( X \) is said to be:

† The point \( x \) is said to be an accumulation point of the set \( X \), if every neighbourhood of \( x \) contains a point of \( X \) other than \( x \). If the space is \( T_1 \), the accumulation points of \( X \) are identical with the points at which \( X \) is not locally finite.

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(1) discrete if \( I(X) = 0 \);
(2) isolated if \( X \cdot I(X) = 0 \);
(3) dense-in-itself if \( X < I(X) \);
(4) scattered, if it contains no dense-in-itself subset;
(5) closed, if \( X = \overline{X} \), or equivalently \( I(X) < X \).

The discrete sets form an ideal \( > I \), which we call \( d \); it is easy to shew that its local function \( d(X) \) is equal to \( I(X) \) (for proof, see IV). At this stage it is necessary to make the additional hypothesis that space is dense-in-itself; in other words that \( I(1) = 1 \). It follows then that every open set as well as every dense set is dense in itself. For if \( G \) is any open set, its complement \( F \) is closed, and \( 1 = I(1) = I(G) + I(F) = G + F \). Since \( I(F) < F \), it follows that \( I(G) > G \) or \( G \) is dense in itself. Again, if \( X \) is a dense set, \( \overline{X} = 1 \), or \( X + I(X) = 1 \), whence \( I(X) + II(X) = I(1) = 1 \). By \( II(d) \) \( II(X) < I(X) \). Hence \( I(X) = 1 > X \); so that \( X \) is dense in itself. It follows from this that scattered sets are non-dense. For if \( X \) is scattered, and \( X \) contains an open set \( G \), then \( G = GX < GX \) \( [II(h)] \), so that \( GX \) is dense in \( G \). But it was shewn that \( G \) is dense-in-itself; hence in the relative topology in which \( G \) is taken as space, \( GX \) being dense must be dense-in-itself. This is a contradiction, as \( X \) being scattered cannot have a dense-in-itself subset \( GX \). Hence \( X \) is non-dense. It follows from this that the scattered sets form an ideal (which we may call \( s \)) containing the ideal \( d \) of discrete sets. For it is clear in the first place that any subset of a scattered set must be scattered. To prove that the union \( X_1 + X_2 \) of two scattered sets \( X_1, X_2 \) is scattered, suppose that \( D \) is a dense-in-itself subset of \( X_1 + X_2 \). Then \( DX_1, DX_2 \) are scattered sets, as subsets of \( X_1, X_2 \). Taking \( D \) as the dense-in-itself space, \( DX_1 \) as scattered set, must be non-dense relatively to \( D \); hence its relative complement, which is a subset of \( DX_2 \), must be dense in \( D \), and therefore dense-in-itself. This contradicts the assumption that \( X_2 \) is scattered. This proves that the union of two scattered sets is scattered. We proceed now to evaluate the local function \( s(X) \) of the ideal \( s \) of scattered sets.

Given any family of dense-in-themselves sets \( X \), it is easy to see that their union \( \Sigma X \) must be dense-in-itself; for \( \Sigma X > \) each \( X \); hence \( I(\Sigma X) > \) each \( I(X) > X \). Hence \( I(\Sigma X) > \Sigma X \), or \( \Sigma X \) is dense-in-itself. If now \( X \) is an arbitrary set, the union \( K(X) \) of all dense-in-themselves subsets of \( X \) must therefore be dense-in-itself; \( K(X) \) is thus the maximal dense-in-itself subset of \( X \), or the dense-in-itself kernel of \( X \). \( X \cdot \{K(X)\}' \) must therefore be a scattered set, since it can have no dense-in-itself subset. We can now see that the local function \( s(X) \) of the ideal \( s \) of scattered sets is equal
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to the closure \( \overline{K(X)} \) of the dense-in-itself kernel \( K(X) \) of \( X \). For if \( x \) be a point of \( \overline{K(X)} \), and \( G \) any neighbourhood of \( x \), \( G \cdot K(X) \neq 0 \), and is a relatively open subset of the dense-in-itself set \( K(X) \). Hence \( G \cdot K(X) \) is dense-in-itself and therefore \( G \cdot X > G \cdot K(X) \) is not scattered for any neighbourhood \( G \) of \( x \). Hence \( X \) is not locally scattered at any point of \( \overline{K(X)} \). On the other hand if \( x \) is any point not in \( \overline{K(X)} \), it has a neighbourhood \( G \) disjoint with \( K(X) \), and \( G \cdot X \) is scattered, since it is a subset of the scattered set \( X \cdot \{K(X)\}' \). Thus \( X \) is locally scattered at every point not in \( \overline{K(X)} \). This proves that \( s(X) = \overline{K(X)} \). It follows from this, that if \( X \cdot s(X) = 0 \), \( X \) must be scattered. For,

\[
0 = X \cdot s(X) = X \cdot \overline{K(X)} > X \cdot K(X) = K(X).
\]

Hence \( K(X) = 0 \), and therefore \( X \) must be scattered.

The next ideal in the series is the ideal \( N \) of non-dense sets (which, as every scattered set is non-dense, contains the ideal \( s \) of scattered sets). A non-dense set is defined to be one whose closure is a boundary set, (or, alternatively, whose exterior is dense). It is obvious from the definition, that a subset of a non-dense set is non-dense. To shew that the non-dense sets form an ideal, we have to shew in addition that the union of two non-dense sets is non-dense. Let \( N_1, N_2 \) be two non-dense sets, and let \( N_1 + N_2 \) contain if possible a non-null open set \( G \). Then

\[
G = GN_1 + GN_2.
\]

Now \( GN_1 \) is a non-dense set, which is also non-dense relative to \( G \); hence its relative complement, which is a subset of \( GN_2 \), would be dense in \( G \), so that its closure would contain the open set \( G \). This contradicts the non-density of \( N_2 \). Thus the existence of the ideal \( N \) of non-dense sets is established. We may shew that the local function \( N(X) \) of this ideal is equal to \( \text{Int.} X \). To prove this, suppose that \( X \) is locally non-dense at \( x \); then there is a neighbourhood \( G \) of \( x \), such that \( GX \), and therefore \( \overline{GX} \) is non-dense. Hence \( GX < \overline{GX} \) must also be non-dense; hence \( G \) must be disjoint with \( \text{Int.} X \), and therefore with \( \overline{\text{Int.} X} \). Thus \( x \) does not belong to \( \overline{\text{Int.} X} \). Conversely if \( x \) does not belong to \( \overline{\text{Int.} X} \), it has a neighbourhood \( G \) disjoint with \( \text{Int.} X \), so that \( \overline{GX} = G \). (Boundary \( X \)) is non-dense, and therefore \( GX < GX \) is non-dense. This establishes the form stated for the local function \( N(X) \).

Lastly, defining a set of the first category as the union of an enumerable family of non-dense sets, it follows at once that the sets of the first category constitute an ideal \( N_\sigma \) which contains \( N \), and is its \( \sigma \)-extension. We shall see presently that \( N_\sigma \) is an example of the supercompact ideal; hence its
local function \( N_\sigma(X) \) has the property \( N_\sigma \{N_\sigma(X)\} = N_\sigma(X) \) (see V below). We can use this to shew that \( N_\sigma(X) \) is a closed domain. For, since \( N_\sigma > N > 0 \), it follows (II (b)) that:

\[
N_\sigma(X) < N(X) = \text{Int. } X < X.
\]

Now \( N_\sigma(X) \) is closed (II (c)); substituting \( N_\sigma(X) \) for \( X \) in this, we have

\[
N_\sigma \text{N}_\sigma(X) < \text{Int. } N_\sigma(X) < N_\sigma(X).
\]

Since \( N_\sigma N_\sigma(X) = N_\sigma(X) \), it follows that \( N_\sigma(X) = \text{Int. } N_\sigma(X) \) is a closed domain. We may set down for reference, the six fundamental ideals and their local functions; viz., \( 0 < I < d < s < N < N_\sigma \).

<table>
<thead>
<tr>
<th>0</th>
<th>I</th>
<th>d</th>
<th>s</th>
<th>N</th>
<th>N_\sigma</th>
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<tr>
<td>The zero-ideal comprising the null-set only</td>
<td>( I(X) = \text{I}(X) )</td>
<td>( d(X) = \text{d}(X) )</td>
<td>( s(X) = \text{s}(X) )</td>
<td>( N(X) = \text{N}(X) )</td>
<td>( N_\sigma(X) = \text{N}_\sigma(X) )</td>
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<td>0(X) = X</td>
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IV. **Compact ideals**.—If \( P \) is any ideal and \( X \in P \), it is clear that \( P(X) = 0 \); but the converse ‘ \( P(X) = 0 \) implies \( X \in P \) ’ is not generally true. If \( P(X) = 0 \) implies \( X \in P \), we shall call \( P \) a compact ideal.

Among the six fundamental ideals listed above, it is easy to see that all are compact with the exception of \( I \), the ideal of finite sets. It is also easy to see that \( I \) will be compact if and only if the given topological space is **compact**, in which case the ideal \( d \) will coincide with \( I \). In fact \( d \) could be described as the minimal compact extension of \( I \). A similar extension can be carried out for a general non-compact ideal \( P \) as follows. Suppose \( Q \) is any compact ideal containing \( P \); then \( Q(X) = 0 \) should imply \( X \in Q \). But as \( Q > P \), \( Q(X) < P(X) \) (II (b)). Hence \( P(X) = 0 \) implies \( Q(X) = 0 \). Hence \( Q \) should contain all sets \( X \) such that \( P(X) = 0 \). But the sets \( X \) such that \( P(X) = 0 \) themselves constitute an ideal (by II (a) and (e)), which we may call \( P_d \). We shall shew that \( P_d \) is compact, so that it is the minimal compact extension of \( P \). We may prove this by shewing that the local function \( P_d(X) \) is equal to \( P(X) \). Suppose that \( X \) is locally \( P_d \) at \( x \); then there is a neighbourhood \( G \) of \( x \), such that \( G \in P \). But as \( G > P \), \( G(X) < P(X) \) (II (b)). Hence \( P(X) = 0 \) implies \( G(X) = 0 \). Hence \( G \) should contain all sets \( X \) such that \( P(X) = 0 \). But the sets \( X \) such that \( P(X) = 0 \) themselves constitute an ideal (by II (a) and (e)), which we may call \( P_d \). We shall shew that \( P_d \) is compact, so that it is the minimal compact extension of \( P \). We may prove this by shewing that the local function \( P_d(X) \) is equal to \( P(X) \). Suppose that \( X \) is locally \( P_d \) at \( x \); then there is a neighbourhood \( G \) of \( x \), such that \( G \in P \). But as \( G > P \), \( G(X) < P(X) \) (II (b)). Hence \( P(X) = 0 \) is equivalent to
\( P(X) = 0 \) which implies by definition that \( X \in P_d \). Thus \( P_d \) is compact and is the minimal compact extension of \( P \).

To explain the significance of this result, we observe that an ideal \( P \) determines not only a local function \( P(X) \) which is the analogue of the derived set, but also a closure function \( X + P(X) \); that \( X + P(X) \) satisfies all Kuratowski's postulates for the closure function follows from \( \Pi(d) \) and \( (e) \). Thus each ideal \( P \) determines, through the closure function \( X + P(X) \), a topology on the space which we may call the \( P \)-topology; since \( X + P(X) < \bar{X} \), the \( P \)-topology is weaker than the original topology, so that open (or closed) sets continue to be open (or closed) in the \( P \)-topology. Also the \( P \)-topology is \( T_1 \), and would be Hausdorff if the original topology is Hausdorff. The ideal \( P_d \) is now seen to consist of the family of discrete sets of the \( P \)-topology. Thus the extension of \( P \) to \( P_d \) is fully parallel to the extension of the ideal of finite sets to the ideal of discrete sets.

V. Supercompact ideals.—We call an ideal \( P \) supercompact, if \( XP(X) = 0 \) implies \( X \in P \). This is a stronger implication than \( 'P(X) = 0 \) implies \( X \in P' \); for if \( XP(X) = 0 \) implies \( X \in P \), then, since \( P(X) = 0 \) implies \( XP(X) = 0 \), it would follow that \( P(X) = 0 \) implies \( X \in P \). Thus a supercompact ideal is necessarily compact, but the converse is not true. For example, the ideal \( d \) of discrete sets is compact but not obviously supercompact. The zero ideal is compact and supercompact. The ideal \( s \) of scattered sets is supercompact, since it was shewn in III that \( Xs(X) = 0 \) implies that \( X \) is scattered. The ideal \( N \) of non-dense sets is supercompact, since \( N(X) = \overline{\text{Int.} X} = 0 \) implies \( \text{Int.} X = 0 \) or \( X \) is non-dense. It will be shewn presently that \( N_s \) is supercompact. The ideal \( 1 \) consisting of all sets of space, is compact and supercompact since its local function \( 1(X) \) is identically zero.

Any one of the following is a necessary and sufficient condition for the ideal \( P \) to be supercompact:

1. \( XP(X) = 0 \) implies \( P(X) = 0 \);
2. \( X \) is locally \( P \) at every one of its points implies \( X \in P \);
3. For every set \( X \), \( X \cdot \{P(X)\}' \in P \).
4. If \( X \) admits a relatively open covering by \( P \)-sets, \( X \in P \).

(1) and (2) only paraphrase the definition. To prove (3), we observe that if \( P \) is supercompact and \( Y = X \cdot \{P(X)\}' \), then
\[
Y \cdot P(Y) = X \cdot \{P(X)\}' \cdot P(X) = X \cdot \{P(X)\}' < X \cdot \{P(X)\}' \cdot P(X) = 0.
\]
Hence \( Y = X \cdot \{P(X)\}' \in P \). Conversely if for all \( X \), \( X \cdot \{P(X)\}' \in P \), then \( Y \cdot P(Y) = 0 \) implies \( Y = Y \cdot \{P(Y) + [P(Y)]'\} = Y \cdot \{P(Y)\}' \in P \). Hence \( P \) is
supercompact. Lastly to prove (4), we have only to observe that $X$ admits a relatively open covering by $P$-sets, if and only if it is locally $P$ at every one of its points.

We have also the important result that if $P$ be supercompact $PP(X) = P(X)$. For if $P$ be supercompact, it follows from (3) that $P \{X(P(X))'\} = 0$. By (g), $P(X) \cdot [PP(X)'] < P\{X \cdot (P(X))'\} = 0$. Hence $P(X) < PP(X)$. But by (d) $PP(X) < P(X)$. Hence the result.

Following a theorem of Banach, we will now shew that,

Any numerical extension of the ideal $N$ is supercompact.

The proof is substantially the same for the general numerical extension, as for the $\sigma$-extension. Let $N_k$ be any numerical extension of $N$ (i.e., the sets of $N_k$ are unions of $N_k$ non-dense sets; $k$ being any ordinal $\geq 0$). Let $S$ be any set admitting a covering by relatively open subsets $X_\alpha$ belonging to $N_\alpha$. We suppose the sets $(X_\alpha)$ to be well-ordered, so that the suffix $\alpha$ runs through the range $1 \leq \alpha < \gamma$. Suppose now we have a well-ordered system $G_\alpha$ of non-null disjoint open sets such that (1) $SG_\alpha N_k$ for all $\alpha$, and (2) the system $S_\alpha$ is saturated. We can now write

$$S = \sum S G_\alpha + (S - \sum G_\alpha).$$

The theorem is proved if we shew that (1) $\sum S G_\alpha N_k$ and (2) $S - \sum G_\alpha N$. To prove (1) write

$$SG_\alpha = \sum N_\beta^\alpha \quad 1 \leq \beta \leq \Omega_k; \quad N_\beta^\alpha \epsilon N.$$

This is possible since $SG_\alpha N_k$. Write now

$$\sum N_\beta^\alpha = N_\beta; \quad \sum SG_\alpha = \sum N_\beta^\alpha (1 \leq \beta \leq \Omega_k);$$

where $\Omega_k$ is the initial ordinal of the class $N_k$. Now any $N_\beta^\alpha$ is relatively open in $N_\beta$, since, the $G_\alpha$'s being mutually disjoint, $N_\beta^\alpha = G_\alpha \cdot N_\beta$. Thus each $N_\beta$ admits a covering by relatively open non-dense subsets, and is therefore non-dense (since the ideal $N$ is supercompact). Therefore $\sum SG_\alpha = \sum N_\beta$ is a set of $N_k$. To prove (2), we have to use the assumed saturation of the system $G_\alpha$. Denote by $F$ the closed set $(\sum G_\alpha)'^\prime$. If $F$ has an interior $H$, then since the system $G_\alpha$ is saturated, $S \cdot H$ does not belong to $N_k$; in particular, $S \cdot H \neq 0$, so that $H$ intersects an $X$, say $X_\tau \cdot HX_\tau$ being a subset of $X_\tau$, belongs to $N_k$. We can find an open set $G < H$ such that $SG = HX_\tau$; namely take $G$ as the part of $H$ contained in $Ext. SX_\tau' = SX_\tau'^\prime$. Since $X_\tau$ is relatively open in $S$, $Ext. SX_\tau'^\prime \cdot S = X_\tau$. Hence $SG = HX_\tau$. We have therefore arrived at the contradiction that there exists a non-null open set
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G, disjoint with all the $G_a$'s, so that $S \cdot G$ belongs to $N_k$. Thus the closed set $F$ can have no interior, and must therefore be non-dense. Hence, $S - \sum G_a$, as a subset of $F$ must also be non-dense.

VI. The supercompact ideal $P_s$.—We have constructed the minimal compact extension $P_d$ (corresponding to the ideal of $P$-discrete sets) of an arbitrary ideal $P$. We will now follow the analogy further, and construct a supercompact extension $P_s$ of $P$, corresponding to the ideal of $P$-scattered sets. It is essential for this extension to assume that space is $P$-dense-in-itself, that is, that $P(1) = 1$.

A set $X$ is $P$-dense-in-itself if $X < P(X)$. It is clear that the union of any family of $P$-dense-in-themselves sets must be $P$-dense-in-itself. Hence if $X$ be any set, the union $K_p(X)$ of all $P$-dense-in-themselves subsets of $X$, is the maximal $P$-dense-in-itself subset of $X$, and may be called the $P$-kernel of $X$.

Any open set of the $P$-topology (and therefore also, any open set) is $P$-dense-in-itself. For if $G$ is open in the $P$-topology, its complement $F$ is closed, so that $P(F) < F$. Hence

$$G + F = 1 = P(1) = P(G) + P(F).$$

Since $P(F) < F$, it follows that $P(G) > G$, or $G$ is $P$-dense-in-itself.

Again any dense set of the $P$-topology is $P$-dense-in-itself.

For if $X$ is dense in the $P$-topology, $X + P(X) = 1$, hence $P(X) + PP(X) = P(1) = 1$. But by II (d), $PP(X) < P(X)$. Hence $P(X) = 1 > X$, so that $X$ is $P$-dense-in-itself.

A set may be said to be $P$-scattered if it contains no $P$-dense-in-itself subset. It follows in particular, that if we remove from a set $X$, its $P$-kernel $K_p(X)$, what remains must be $P$-scattered. Any $P$-scattered set is $P$-non-dense. For let $X$ be $P$-scattered, and let its $P$-closure $X + P(X)$ contain a $P$-open set $G$. Then $G = G$. $P$-closure of $X < P$-closure of $GX$. Hence $GX$ is dense in $G$ (in the $P$-topology). Consider the relative $P$-topology in which $G$ is taken as space; the condition that space is dense-in-itself is satisfied in this topology, since as a $P$-open set is $P$-dense-in-itself. $GX$ being dense in this space must be $P$-dense-in-itself. This contradicts the assumption that $X$, as scattered set contains no $P$-dense-in-itself subset. Hence the theorem.

It is clear that any subset of a $P$-scattered set is $P$-scattered. Also the union $X_1 + X_2$ of two $P$-scattered sets must be $P$-scattered. For if it contains a $P$-dense-in-itself subset $D$, then $DX_1$ must be non-dense in the relative $P$-topology of $D$, hence its relative complement, which is a subset of $DX_2$,
must be dense in D, and therefore P-dense-in-itself, contradicting the assumption that $X_2$ is P-scattered. Thus the P-scattered sets form an ideal $P$ which contains $P_d$.

We can now shew that the local function $P_s(X)$ of the ideal $P_s$ is equal to $K_p(X)$, where $K_p(X)$ is the P-kernel of $X$. For if $x$ is a point belonging to the kernel $K_p(X)$, and $G$ is any open neighbourhood of $x$, $GX > G \cdot K_p(X) \neq 0$. Now $G$ is open and therefore open in the P-topology also. Considering $K_p(X)$ as the dense-in-itself of its relative P-topology, $G \cdot K_p(X)$ as relatively open subset of $K_p(X)$ is P-dense-in-itself. Thus $GX$ is not P-scattered; or $X$ is not locally P-scattered at $x$. On the other hand if $x$ is not in $K_p(X)$, it has an open neighbourhood $G$ disjoint with $K_p(X)$; then $GX$ is a subset of the P-scattered set $X \cdot \{K_p(X)\}'$, and therefore P-scattered. Thus $X$ is locally P-scattered at $x$. This proves that $P_s(X) = K_p(X)$.

We can see finally that $P_s$ is supercompact, so that it is a super-compact extension of $P$ or $P_d$. For if $XP_s(X) = 0$, then

$$K_p(X) = XK_p(X) < X \cdot K_p(X) = X \cdot P_s(X) = 0;$$

hence $X$ is P-scattered and belongs to $P_s$, since its P-kernel is null.

VII. The supercompact ideal $P_N$.—The non-dense sets of the P-topology, form an ideal $P_N$ which contains the ideal $P_s$ of the P-scattered sets. This does not require a special proof; for, so long as we are handling only a single topology, e.g., the P-topology, general theorems like ‘Non-dense sets constitute an ideal’ will continue to be true. It is only in the matter of the local functions that we have to exercise care, since there is a mix-up of two topologies (the original topology of the space enters through the neighbourhoods used in the definition of the local function).

We next proceed to show that the local function $P_N(X)$ of $P_N$ is equal to closure. Int$_p$·closure$_p$ (X) = Int$_p$ (X + P (X))

(where ‘Int$_p$’ means that the Interior function is to be interpreted in the sense of the P-topology). To prove this, let $X$ be locally P-non-dense at $x$; then there is an open neighbourhood $G$ of $x$, so that $GX$ is P-non-dense, and therefore also closure$_p$ (GX) is P-non-dense. Now $G$ is an open set of the original topology, and therefore an open set of the P-topology; hence we can use the formula $GX < \overline{GX}$; hence

$$G \cdot \text{closure}_p(X) < \text{closure}_p(GX),$$

and is therefore P-non-dense. Hence $G$ must be disjoint with Int$_p$·closure$_p$ (X); since $G$ is an open set of the original topology, it follows that $x$ does not belong to closure·Int$_p$·closure$_p$(X) = Int$_p$(X + P (X)). Conversely, if $x$
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does not belong to $\text{Int}_p (X + P (X))$, it follows that it has an open neighbourhood $G$ disjoint with $\text{Int}_p (X + P (X))$. Hence $GX < G (X + P (X)) = G$. Boundary$_p (X + P (X))$. Since the boundary of a closed set is non-dense, it follows that the last term, and hence $GX$ is $P$-non-dense. Hence $X$ is locally $P$-non-dense at $x$. This establishes the form stated for the local function $P_N (X)$.

Finally, we can shew that the ideal $P_N$ is supercompact, and therefore a supercompact extension of the supercompact ideal $P$. For, if $X \cdot P_N (X) = 0$, $X \cdot \text{Int}_p \cdot \{\text{closure}_p X\} < X \cdot P_N (X)$ is also null. Since $X < \text{closure}_p (X)$, and since $X$ is disjoint with $\text{Int}_p \cdot \{\text{closure}_p (X)\}$, it follows that $X < \text{Boundary}_p \{\text{closure}_p X\}$. Since the boundary of a closed set is non-dense, it follows that $X$ is $P$-non-dense, and therefore belongs to $P_N$. Thus $P_N$ is supercompact.

VIII. The ideal $P_{N\sigma}$.—The sets of the first category in the $P$-topology form the $\sigma$-extension of the ideal $P_N$. We would not however be able to say that this extension is supercompact; for an examination of the proof of the supercompactness of numerical extensions of $N$ will shew that the non-dense character enters essentially in the proof. For a similar reason, we cannot assert the supercompactness of a numerical extension of any supercompact ideal.