THE LOCALISATION THEORY IN SET-TOPOLOGY

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The earliest notion which could be described as the starting point of set-topology is Cantor's concept of the derived set D(X) of a set X of real numbers, defined as the set of points of X which are accumulation points thereof. This was generalised by Frechêt into the abstract topological notion of the *derived set* D(X) of a set X, defined as the set of all *limit-elements* of X. In the present-day treatments of set-topology, a greater prominence is generally given to the *closure function* $\bar{X} = X + D(X)$, introduced postulationally by Kuratowski. The connection between the *closure* and the *derivate* of a set X may be described by saying that in the topology defined by the closure-function, both the closure $\bar{X}$ and the derived set D(X) appear as *local* functions of X ; namely $\bar{X}$ is the set of points at which X is not locally *null,* while (assuming that the topology is T_1) D(X) is the set of points at which X is not locally *finite.* The idea of localisation of properties thus suggested has been treated in a general manner by Kuratowski in his *Topologie,* with a systematic calculus; a notable achievement of the calculus is the elegant proof that it gives of the theorem that *the points at which a set of the second category is* **locally** *of the second category constitute a closed domain.* (The corresponding theorem for metric spaces was originally stated and proved by Banach.)

In this paper, I review the localisation theory and study the properties of what I have called *compact* and *super-compact* ideals (or hereditary additive properties), as well as certain extensions P_α, P_S, P_N of an ideal P.

I. Let R be a topological T_1-space, B_R the boolean algebra of all its subsets. A property P of subsets of R is *hereditary* if $Y < X$ and $X \epsilon P$ imply $Y \epsilon P$; it is *additive,* if $X \epsilon P$ and $Y \epsilon P$ imply $X + Y \epsilon P$. Given any property P of subsets of R, it is convenient to denote also by P the family of all subsets possessing the property P. If P is a hereditary additive property, it is clear that the family P is a μ-ideal of B_R; conversely corresponding to any μ-ideal P of B_R we have a hereditary additive property P, *viz.,* the property of belonging to the ideal P. In particular if P is the zero ideal, *i.e.,* the ideal containing the null set only, the corresponding hereditary additive property is the property of being null; if P is the ideal 1, *i.e.,* the whole of B_R, the corresponding property is the universal property of being a subset of R; and so on.

We say that a set X has the property P *locally* at a point x, if there exists a neighbourhood U_x of x, such that $U_x \cdot X \epsilon P$. The set of points at which X does *not* have the property P locally is denoted by P (X). Thus $x \epsilon P(X)$ means that for *every* neighbourhood U_x of x, $U_x \cdot X$ does not have the property P, or briefly is not a P. If P is a hereditary property it will be sufficient to restrict ourselves to *open* neighbourhoods; for if $U_x \cdot X \epsilon P$, $(\text{Int. } U_x) \cdot X < U_x \cdot X$ is also a P, and since Int. U_x is also neighbourhood, $(\text{Int. } U_x) \cdot X \epsilon P$ implies X is locally P at x.

We shall consider only hereditary additive properties P; we shall call P (X) the *local function* of the corresponding ideal P.

II. *General Properties of the local function* P (X).*

(*a*) If $X < Y$, $P(X) < P(Y)$.

For if p is not in P (Y), it has an open neighbourhood G_p such that $G_p \cdot Y \epsilon P$.

Then $G_p \cdot X < G_p \cdot Y$ is also a P. Hence p is not in P (X). Thus $\{P(Y)\}' < \{P(X)\}'$ or $P(X) < P(Y)$.

(*b*) If $P < Q$ (*i.e.*, if the *ideal* P is contained in the ideal Q), $P(X) > Q(X)$.

For if X is locally P at x, it must also be locally Q at x (since every P is also a Q). Hence $\{P(X)\}' < \{Q(X)\}'$ or $P(X) > Q(X)$.

(*c*) P (X) is a closed set contained in $\bar{X}$.

For $\bar{X}$ is the local function O (X) of the zero ideal. Since the ideal P contains the zero ideal, it follows from (*b*) that $P(X) < \bar{X}$. To shew that P (X) is closed, any point $p \epsilon \{P(X)\}'$ has an open neighbourhood G_p such that $G_p \cdot X \epsilon P$. It is clear that X is locally P at every point of G_p. Thus $G_p < \{P(X)\}'$, or $\{P(X)\}'$ is open. Hence P (X) is closed.

(*d*) $PP(X) < P(X)$.

For by (*c*) $PP(X) < \overline{P(X)} = P(X)$.

(*e*) $P(X + Y) = P(X) + P(Y)$.

From (*a*) it follows that $P(X + Y) > P(X) + P(Y)$. To prove the reverse inclusion, let p belong neither to P (X) nor to P (Y). Therefore it has open neighbourhoods U_p, V_p, so that $U_p \cdot X \epsilon P$, $V_p \cdot Y \epsilon P$. As P is here-

*All these properties with the exception of (*b*) and (*f*) will be found in Kuratowski's *Topologie*, pp. 29, 30.

ditary and additive $U_pV_p(X+Y)\,\epsilon P$. Since U_pV_p is an open neighbourhood of p, it follows that p does not belong to $P(X+Y)$. Hence $\{P(X)\}'\cdot\{P(Y)\}' < \{P(X+Y)\}'$ or $P(X+Y) < P(X) + P(Y)$. Hence the result.

(*f*) If P_3 is the intersection of the ideals P_1, P_2, $P_3(X) = P_1(X) + P_2(X)$.

For by (*b*) $P_3(X) > P_1(X) + P_2(X)$. To prove the reverse inclusion, we observe that if X is locally P_1 as well as locally P_2 at x, there must exist open neighbourhoods U_x, V_x of x, so that $U_x\cdot X\epsilon P_1$, $V_x\cdot X\epsilon P_2$. Hence U_xV_xX is a P_1 and also a P_2; that is, $U_xV_xX\epsilon P_3$ or X is locally P_3 at x. Hence if X is *not* locally P_3 at x, it is either not locally P_1 or not locally P_2 there; *i.e.*, $P_3(X) < P_1(X) + P_2(X)$. This proves that $P_3(X) = P_1(X) + P_2(X)$.

(*g*) $P(X)\{P(Y)\}' = P(XY')\{P(Y)\}' < P(XY')$.

This follows since $P(X) = P(XY) + P(XY')$ and $P(XY) < P(Y)$ by (*a*).

(*h*) If G is open $G\cdot P(X) = G\cdot P(GX) < P(GX)$.

For if $p\epsilon G\cdot P(X)$, and H any open neighbourhood of p, HG is also a neighbourhood, and therefore HGX is not a P. Hence $p\epsilon P(GX)$. Thus $G\cdot P(X) < P(GX) < P(X)$ by (*a*).

Hence $GP(X) < G\cdot P(GX) < GP(X)$, which proves (*h*).

The additive property of P has been used in proving (*e*), (*g*) only.

Hence the remaining properties are all valid if P is hereditary without being additive.

III. *The fundamental series of ideals and their local functions.*—It was already noticed that the closure function $\bar{X}$ is the local function of the zero ideal O. This is the lowest of a series of ideals, which we may call *numerical ideals,* and denote by I_ω; where for any ordinal $\omega \geqslant 0$, we denote by I_ω the ideal of all sets whose potency is less than $\aleph_\omega$. I_0 (which we may write simply I) is thus the ideal of *finite* sets, I_1 the ideal of sets which are either finite or enumerable and so on. O and I $(=I_0)$ are the basic ideals we have to consider; the local function O(X) is $\bar{X}$, while the local function I(X) is the *derived set* of X, defined as the set of all accumulation points of X.† From II (*c*) I(X) is a closed set contained in $\bar{X}$, and $\bar{X} = X + I(X)$. A set X is said to be:

† The point x is said to be an *accumulation point* of the set X, if every neighbourhood of x contains a point of X other than x. If the space is T_1, the accumulation points of X are identical with the points at which X is not locally finite.

(1) *discrete* if $I(X) = 0$;

(2) *isolated* if $X \cdot I(X) = 0$;

(3) *dense-in-itself* if $X < I(X)$;

(4) *scattered,* if it contains no dense-in-itself subset;

(5) *closed,* if $X = \overline{X}$, or equivalently $I(X) < X$.

The discrete sets form an ideal $> I$, which we call d; it is easy to shew that its local function $d(X)$ is equal to $I(X)$ (for proof, see IV). At this stage it is necessary to make the additional hypothesis that *space is dense-in-itself*; in other words that $I(1) = 1$. It follows then that *every open set as well as evey dense set is dense in itself.* For if G is any open set, its complement F is closed, and $1 = I(1) = I(G) + I(F) = G + F$. Since $I(F) < F$, it follows that $I(G) > G$ or G is dense in itself. Again, if X is a dense set, $\overline{X} = 1$, or $X + I(X) = 1$, whence $I(X) + II(X) = I(1) = 1$. By II (d) $II(X) < I(X)$. Hence $I(X) = 1 > X$; so that X is dense in itself. It follows from this that *scattered sets are non-dense.* For if X is scattered, and $\overline{X}$ contains an open set G, then $G = G\overline{X} < \overline{GX}$ [II (h)], so that GX is dense in G. But it was shewn that G is dense-in-itself; hence in the relative topology in which G is taken as space, GX being dense must be dense-in-itself. This is a contradiction, as X being scattered cannot have a dense-in-itself subset GX. Hence X is non-dense. It follows from this that *the scattered sets form an ideal* (*which we may call* **s**) *containing the ideal* **d** *of discrete sets.* For it is clear in the first place that any subset of a scattered set must be scattered. To prove that the union $X_1 + X_2$ of two scattered sets X_1, X_2 is scattered, suppose that D is a dense-in-itself subset of $X_1 + X_2$. Then DX_1, DX_2 are scattered sets, as subsets of X_1, X_2. Taking D as the dense-in-itself space, DX_1 as scattered set, must be non-dense relatively to D; hence its relative complement, which is a subset of DX_2, must be dense in D, and therefore dense-in-itself. This contradicts the assumption that X_2 is scattered. This proves that the union of two scattered sets is scattered. We proceed now to evaluate the local function $s(X)$ of the ideal s of scattered sets.

Given any family of dense-in-themselves sets X, it is easy to see that their union ΣX must be dense-in-itself; for $\Sigma X >$ each X; hence $I(\Sigma X) >$ each $I(X) > X$. Hence $I(\Sigma X) > \Sigma X$, or ΣX is dense-in-itself. If now X is an arbitrary set, the union $K(X)$ of all dense-in-themselves subsets of X must therefore be dense-in-itself; $K(X)$ is thus the *maximal* dense-in-itself subset of X, or *the dense-in-itself kernel* of X. $X \cdot \{K(X)\}'$ must therefore be a scattered set, since it can have no dense-in-itself subset. We can now see that *the local function* $s(X)$ *of the ideal* **s** *of scattered sets is equal* *to the closure $\overline{K(X)}$ of the dense-in-itself kernel $K(X)$ of X.* For if x be a point of $\overline{K(X)}$, and G *any* neighbourhood of x, $G.K(X) \neq 0$, and is a relatively open subset of the dense-in-itself set K (X). Hence $G \cdot K(X)$ is dense-in-itself and therefore $G \cdot X > G \cdot K(X)$ is not scattered for any neighbourhood G of x. Hence X is not locally scattered at any point of $\overline{K(X)}$. On the other hand if x is any point not in $\overline{K(X)}$, it has a neighbourhood G disjoint with K (X), and $G \cdot X$ is scattered, since it is a subset of the scattered set $X \cdot \{K(X)\}'$. Thus X is locally scattered at every point not in $\overline{K(X)}$. This proves that $s(X) = \overline{K(X)}$. It follows from this, that if $X \cdot s(X) = 0$, X must be scattered. For,

$$0 = X \cdot s(X) = X \cdot \overline{K(X)} > X \cdot K(X) = K(X).$$

Hence $K(X) = 0$, and therefore X must be scattered.

The next ideal in the series is *the ideal N of non-dense sets* (which, as every scattered set is non-dense, contains the ideal s of scattered sets). A *non-dense set* is defined to be one whose closure is a boundary set, (or, alternatively, whose exterior is dense). It is obvious from the definition, that a subset of a non-dense set is non-dense. To shew that the non-dense sets form an ideal, we have to shew in addition that the union of two non-dense sets is non-dense. Let N_1, N_2 be two non-dense sets, and let $\overline{N}_1 + \overline{N}_2$ contain if possible a non-null open set G. Then

$$G = G\overline{N}_1 + G\overline{N}_2.$$

Now $G\overline{N}_1$ is a non-dense set, which is also non-dense relative to G; hence its relative complement, which is a subset of $G\overline{N}_2$, would be dense in G, so that its closure would contain the open set G. This contradicts the non-density of N_2. Thus the existence of the ideal N of non-dense sets is established. We may shew that *the local function $N(X)$ of this ideal is equal to $\overline{Int. \overline{X}}$.* To prove this, suppose that X is locally non-dense at x; then there is a neighbourhood G of x, such that GX, and therefore $\overline{GX}$ is non-dense. Hence $G\overline{X} < \overline{GX}$ must also be non-dense; hence G must be disjoint with Int. $\overline{X}$, and therefore with $\overline{Int. \overline{X}}$. Thus x does not belong to $\overline{Int. \overline{X}}$. Conversely if x does not belong to $\overline{Int. \overline{X}}$, it has a neighbourhood G disjoint with Int. $\overline{X}$, so that $G\overline{X} = G$. (Boundary $\overline{X}$) is non-dense, and therefore $GX < G\overline{X}$ is non-dense. This establishes the form stated for the local function N (X).

Lastly, defining *a set of the first category* as the union of an enumerable family of non-dense sets, it follows at once that the sets of the first category constitute an ideal N_σ which contains N, and is its σ-extension. We shall see presently that N_σ is an example of the supercompact ideal; hence its local function $N_\sigma(X)$ has the property $N_\sigma\{N_\sigma(X)\} = N_\sigma(X)$ (see V below). We can use this to shew that $N_\sigma(X)$ is *a closed domain.* For, since $N_\sigma > N > 0$, it follows (II (*b*)) that:

$$N_\sigma(X) < N(X) = \overline{\text{Int.}\,\overline{X}} < \overline{X}.$$

Now $N_\sigma(X)$ is closed (II (*c*)); substituting $N_\sigma(X)$ for X in this, we have

$$N_\sigma N_\sigma (X) < \overline{\text{Int.}\,\overline{N_\sigma(X)}} < N_\sigma(X).$$

Since $N_\sigma N_\sigma(X) = N_\sigma(X)$, it follows that $N_\sigma(X) = \overline{\text{Int.}\,\overline{N_\sigma(X)}}$ is a closed domain. We may set down for reference, the six fundamental ideals and their local functions; *viz.*, $0 < I < d < s < N < N_\sigma$.

0	I	*d*	*s*	N	N_σ
The zero-ideal comprising the null-set only $0(X) = \overline{X}$	The ideal of finite sets $I(X)$ = derived set of X	The ideal of discrete sets $d(X) = I(X)$	The ideal of scattered sets $s(X) = \overline{K(X)}$ K (X)=dense-in-itself kernel of X	The ideal of non-dense sets $N(X) = \overline{\text{Int.}\,\overline{X}}$	The ideal of sets of the 1st category $N_\sigma(X)$= closed domain < N (X)

IV. *Compact ideals.*—If P is any ideal and $X \epsilon P$, it is clear that $P(X) = 0$; but the converse ' $P(X) = 0$ implies $X \epsilon P$ ' is not generally true. If $P(X) = 0$ implies $X \epsilon P$, we shall cell P a *compact* ideal.

Among the six fundamental ideals listed above, it is easy to see that all are compact with the exception of I, the ideal of finite sets. It is also easy to see that I will be compact if and only if the given topological space is *compact,* in which case the ideal *d* will coincide with I. In fact *d* could be described as the minimal compact extension of I. A similar extension can be carried out for a general non-compact ideal P as follows. Suppose Q is any compact ideal containing P; then $Q(X) = 0$ should imply $X \epsilon Q$. But as $Q > P$, $Q(X) < P(X)$ (II (*b*)). Hence $P(X) = 0$ implies $Q(X) = 0$. Hence Q should contain all sets X such that $P(X) = 0$. But the sets X such that $P(X) = 0$ themselves constitute an ideal (by II (*a*) and (*e*)), which we may call P_d. We shall shew that P_d is compact, so that it is the *minimal compact extension* of P. We may prove this by shewing that the local function $P_d(X)$ is equal to $P(X)$. Suppose that X is locally P_d at x; then there is a neighbourhood G of x, such that $GX \epsilon P_d$, that is, such that $P(GX) = 0$. It follows that x is not in $P(GX)$, and therefore GX is locally P at x; hence there is a neighbourhood G' of x, so that $G'GX \epsilon P$. As $G'G$ is itself a neighbourhood of x, this shews that X is locally P at x. Hence $\{P_d(X)\}' < \{P(X)\}'$, or $P_d(X) > P(X)$. But $P_d > P$, so that by II (*b*) $P_d(X) < P(X)$. Hence $P_d(X) = P(X)$. Hence $P_d(X) = 0$ is equivalent to

$P(X) = 0$ which implies by definition that $X \epsilon P_d$. Thus P_d is compact and is the minimal compact extension of P.

To explain the significance of this result, we observe that an ideal P determines not only a local function $P(X)$ which is the analogue of *the derived set*, but also a closure function $X + P(X)$; that $X + P(X)$ satisfies all Kuratowski's postulates for the closure function follows from II (*d*) and (*e*). Thus each ideal P determines, through the closure function $X + P(X)$, a topology on the space which we may call the P-topology; since $X + P(X) < \bar{X}$, the P-topology is *weaker* than the original topology, so that open (or closed) sets continue to be open (or closed) in the P-topology. Also the P-topology is T_1, and would be Hausdorff if the original topology is Hausdorff. The ideal P_d is now seen to consist of the family of *discrete sets* of the P-topology. Thus the extension of P to P_d is fully parallel to the extension of the ideal of finite sets to the ideal of discrete sets.

V. *Supercompact ideals.*—We call an ideal P *supercompact*, if $XP(X) = 0$ implies $X \epsilon P$. This is a stronger implication than '$P(X) = 0$ implies $X \epsilon P$'; for if $XP(X) = 0$ implies $X \epsilon P$, then, since $P(X) = 0$ implies $XP(X) = 0$, it would follow that $P(X) = 0$ implies $X \epsilon P$. Thus a supercompact ideal is necessarily compact, but the converse is not true. For example, the ideal *d* of discrete sets is compact but not obviously supercompact. The zero ideal is compact and supercompact. The ideal *s* of scattered sets is supercompact, since it was shewn in III that $Xs(X) = 0$ implies that X is scattered. The ideal N of non-dense sets is supercompact, since $N(X) = \overline{\text{Int.}\, \bar{X}} = 0$ implies $\text{Int.}\, \bar{X} = 0$ or X is non-dense. It will be shewn presently that N_σ is supercompact. The ideal 1 consisting of all sets of space, is compact and supercompact since its local function $1(X)$ is identically zero.

Any one of the following is a necessary and sufficient condition for the ideal P to be supercompact:

(1) $XP(X) = 0$ implies $P(X) = 0$;

(2) X is locally P at every one of its points implies $X \epsilon P$;

(3) For every set X, $X \cdot \{P(X)\}' \epsilon P$,

(4) If X admits a relatively open covering by P-sets, $X \epsilon P$.

(1) and (2) only paraphrase the definition. To prove (3), we observe that if P is supercompact and $Y = X \cdot \{P(X)\}'$, then

$$Y \cdot P(Y) = X \cdot \{P(X)\}' \cdot P\{X \cdot (P(X)'\} < X \cdot \{P(X)\}' \cdot P(X) = 0.$$

Hence $Y = X \cdot \{P(X)\}' \epsilon P$. Conversely if for all X, $X \cdot \{P(X)\}' \epsilon P$, then $Y \cdot P(Y) = 0$ implies $Y = Y\{P(Y) + [P(Y)]'\} = Y\{P(Y)\}' \epsilon P$. Hence P is supercompact. Lastly to prove (4), we have only to observe that X admits a relatively open covering by P-sets, if and only if it is locally P at every one of its points.

We have also the important result that *if P be supercompact $PP(X) = P(X)$*. For if P be supercompact, it follows from (3) that $P(X\{P(X)\}') = 0$. By II (*g*), $P(X)\cdot[PP(X)]' < P\{X\cdot(P(X))'\} = 0$. Hence $P(X) < PP(X)$. But by II (*d*) $PP(X) < P(X)$. Hence the result.

Following a theorem of Banach, we will now shew that,

Any numerical extension of the ideal N is supercompact.

The proof is substantially the same for the general numerical extension, as for the σ-extension. Let N_k be any numerical extension of N (*i.e.*, the sets of N_k are unions of $\aleph_k$ non-dense sets; k being any ordinal $\geqslant 0$). Let S be any set admitting a covering by relatively open subsets X_α belonging to N_k. We suppose the sets (X_α) to be well-ordered, so that the suffix α runs through the range $1 \leqslant \alpha < \gamma$. Suppose now we have a well-ordered system G_α of non-null disjoint open sets such that (1) $SG_\alpha \epsilon N_k$ for all α, and (2) the system S_α is saturated. We can now write

$$S = \sum_\alpha S\,G_\alpha + (S - \Sigma\,G_\alpha).$$

The theorem is proved if we shew that (1) $\sum_\alpha S\,G_\alpha \epsilon\, N_k$ and (2) $S - \Sigma\,G_\alpha \epsilon N$. To prove (1) write

$$SG_\alpha = \sum_\beta N_\beta{}^\alpha \quad 1 \leqslant \beta \leqslant \Omega_k;\; N_\beta{}^\alpha \,\epsilon\, N.$$

This is possible since $SG_\alpha \epsilon N_k$. Write now

$$\sum_\alpha N_\beta{}^\alpha = N_\beta;\; \sum_\alpha SG_\alpha = \sum_\beta N_\beta\cdot(1 \leqslant \beta \leqslant \Omega_k);$$

where Ω_k is the initial ordinal of the class $\aleph_k$. Now any $N_\beta{}^\alpha$ is relatively open in N_β, since, the G_α's being mutually disjoint, $N_\beta{}^\alpha = G_\alpha\cdot N_\beta$. Thus each N_β admits a covering by relatively open non-dense subsets, and is therefore non-dense (since the ideal N is supercompact). Therefore $\Sigma\, SG_\alpha = \Sigma N_\beta$ is a set of N_k. To prove (2), we have to use the assumed saturation of the system G_α. Denote by F the closed set $(\Sigma G_d)'$. If F has an interior H, then since the system G_α is saturated, SH does not belong to N_k; in particular, $S\cdot H \neq 0$, so that H intersects an X, say $X_t\cdot HX_t$ being a subset of X_t, belongs to N_k. We can find an open set $G < H$ such that $SG = HX_t$; namely take G as the part of H contained in Ext. $SX_t' = \overline{SX_{t'}}'$. Since X_t is relatively open in S, Ext. $SX_t'\cdot S = X_t$. Hence $SG = HX_t$. We have therefore arrived at the contradiction that there exists a non-null open set

G, disjoint with all the G_α's, so that $S \cdot G$ belongs to N_k. Thus the closed set F can have no interior, and must therefore be non-dense. Hence, $S - \Sigma G_\alpha$, as a subset of F must also be non-dense.

VI. *The supercompact ideal P_s.*—We have constructed the minimal compact extension P_d (corresponding to the ideal of P-discrete sets) of an arbitrary ideal P. We will now follow the analogy further, and construct a supercompact extension P_s of P, corresponding to the ideal of P-scattered sets. It is essential for this extension to assume that space is P-dense-in-itself, that is, that $P(1) = 1$.

A set X is *P-dense-in-itself* if $X < P(X)$. It is clear that the union of any family of P-dense-in-themselves sets must be P-dense-in-itself. Hence if X be any set, the union $K_p(X)$ of all P-dense-in-themselves subsets of X, is the maximal P-dense-in-itself subset of X, and may be called the P-kernel of X.

Any open set of the P-topology (and therefore also, any open set) is P-dense-in-itself. For if G is open in the P-topology, its complement F is closed, so that $P(F) < F$. Hence

$$G + F = 1 = P(1) = P(G) + P(F).$$

Since $P(F) < F$, it follows that $P(G) > G$, or G is P-dense-in-itself.

Again any dense set of the P-topology is P-dense-in-itself.

For if X is dense in the P-topology, $X + P(X) = 1$; hence $P(X) + PP(X) = P(1) = 1$. But by II (*d*), $PP(X) < P(X)$. Hence $P(X) = 1 > X$, so that X is P-dense-in-itself.

A set may be said to be *P-scattered* if it contains no P-dense-in-itself subset. It follows in particular, that if we remove from a set X, its P-kernel $K_p(X)$, what remains must be P-scattered. *Any P-scattered set is P-non-dense.* For let X be P-scattered, and let its P-closure $X + P(X)$ contain a P-open set G. Then $G = G$. P-closure of $X <$ P-closure of GX. Hence GX is dense in G (in the P-topology). Consider the relative P-topology in which G is taken as space; the condition that space is dense-in-itself is satisfied in this topology, since as a P-open set is P-dense-in-itself. GX being dense in this space must be P-dense-in-itself. This contradicts the assumption that X, as scattered set contains no P-dense-in-itself subset. Hence the theorem.

It is clear that any subset of a P-scattered set is P-scattered. Also *the union $X_1 + X_2$ of two P-scattered sets must be P-scattered.* For if it contains a P-dense-in-itself subset D, then DX_1 must be non-dense in the relative P-topology of D, hence its relative complement, which is a subset of DX_2, must be dense in D, and therefore P-dense-in-itself, contradicting the assumption that X_2 is P-scattered. Thus *the P-scattered sets form an ideal P_s which contains P_d.*

We can now shew that *the local function $P_s(X)$ of the ideal P_s, is equal to $\overline{K_p(X)}$, where $K_p(X)$ is the P-kernel of X.* For if x is a point belonging to the kernel $\overline{K_p(X)}$, and G is any open neighbourhood of x, $GX > G \cdot K_p(X) \neq 0$. Now G is open and therefore open in the P-topology also. Considering $K_p(X)$ as the dense-in-itself of its relative P-topology, $G \cdot K_p(X)$ as relatively open subset of $K_p(X)$ is P-dense-in-itself. Thus GX is not P-scattered; or X is not locally P-scattered at x. On the other hand if x is not in $K_p(X)$, it has an open neighbourhood G disjoint with $K_p(X)$; then GX is a subset of the P-scattered set $X \cdot \{K_p(X)\}'$, and therefore P-scattered. Thus X is locally P-scattered at x. This proves that $P_s(X) = \overline{K_p(X)}$.

We can see finally that P_s is supercompact, so that it is a super-compact extension of P or P_d. For if $XP_s(X) = 0$, then

$$K_p(X) = XK_p(X) < X \cdot \overline{K_p(X)} = X \cdot P_s(X) = 0;$$

hence X is P-scattered and belongs to P_s, since its P-kernel is null.

VII. *The supercompact ideal P_N.*—The non-dense sets of the P-topology, form an ideal P_N which contains the ideal P_s of the P-scattered sets. This does not require a special proof; for, so long as we are handling only a single topology, *e.g.*, the P-topology, general theorems like 'Non-dense sets constitute an ideal' will continue to be true. It is only in the matter of the local functions that we have to exercise care, since there is a mix-up of two topologies (the original topology of the space enters through the neighbourhoods used in the definition of the local function).

We next proceed to shew that *the local function $P_N(X)$ of P_N is equal to*

$$\text{closure. } \mathrm{Int}_p \cdot \mathrm{closure}_p (X) = \overline{\mathrm{Int}_p (X + P(X))}:$$

(where 'Int_p' means that the Interior function is to be interpreted in the sense of the P-topology). To prove this, let X be locally P-non-dense at x; then there is an open neighbourhood G of x, so that GX is P-non-dense, and therefore also $\mathrm{closure}_p(GX)$ is P-non-dense. Now G is an open set of the original topology, and therefore an open set of the P-topology; hence we can use the formula $G\overline{X} < \overline{GX}$; hence

$$G \cdot \mathrm{closure}_p (X) < \mathrm{closure}_p (GX),$$

and is therefore P-non-dense. Hence G must be disjoint with $\mathrm{Int}_p \, \mathrm{closure}_p (X)$; since G is an open set of the original topology, it follows that x does not belong to $\mathrm{closure} \cdot \mathrm{Int}_p \; \mathrm{closure}_p(X) = \overline{\mathrm{Int}_p(X + P(X))}$. Conversely, if x

does not belong to Int.$_p$ (X + P (X)), it follows that it has an open neighbourhood G disjoint with Int$_p$ (X + P (X)). Hence GX < G (X + P (X)) = G. Boundary$_p$ (X + P (X)). Since the boundary of a closed set is non-dense, it follows that the last term, and hence GX is P-non-dense. Hence X is locally P-non-dense at x. This establishes the form stated for the local function P_N (X).

Finally, we can shew that the ideal P_N is supercompact, and therefore a supercompact extension of the supercompact ideal P_s. For, if X·P_N(X) = 0, X·Int.$_p$·{closure$_p$ X) < X · P_N(X) is also null. Since X < closure$_p$ (X), and since X is disjoint with Int.$_p$·{closure$_p$ (X)}, it follows that X < Boundary$_p$ {closure$_p$ X}. Since the boundary of a closed set is non-dense, it follows that X is P-non-dense, and therefore belongs to P_N. Thus P_N is supercompact.

VIII. *The ideal $P_{N\sigma}$.*—The sets of the first category in the P-topology form the σ-extension of the ideal P_N. We would not however be able to say that this extension is supercompact; for an examination of the proof of the supercompactness of numerical extensions of N will shew that the non-dense character enters essentially in the proof. For a similar reason, we cannot assert the supercompactness of a numerical extension of any supercompact ideal.